Lesson 1

Numbers, sets, and induction

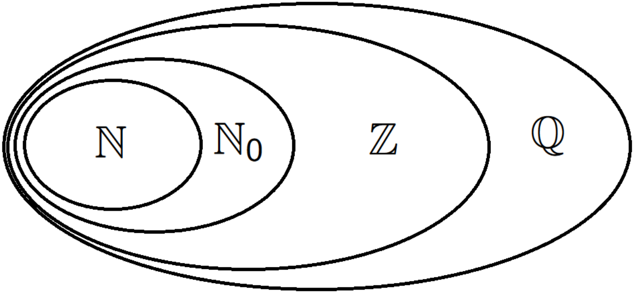
MATH 311, Section 4, FALL 2022

September 6th, 2022

Read Chapter 1 in Gaughan's book.

Number systems

$\mathbb{N} = \{1, 2, 3, \ldots\}$- positive integers,
$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ - non-negative integers,
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$ - the set of integers,
$\mathbb{Q} = \{\frac{m}{n},\ m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}\}$ - the set of rationals.



Sets

The words **family** and **collection** will be used synonymously with "set".

Notation

$\emptyset$ - **empty set**,
$\mathcal{P}(X)$ - family of subsets of the set X, sometimes called **power set** of X.

Example 1

If $X = \{1\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{1\}\}.$$

Example 2

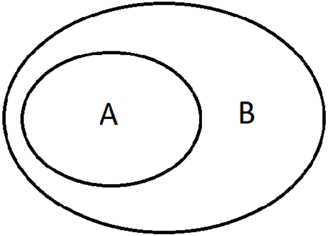
If $X = \{1, 2, 3\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Inclusions 1/2

Definition (Inclusion in a weak sense)

We write $A \subseteq B$ if any element of A is also the element of B.



We will write $A \subset B$ if $A \subseteq B$ and $A \neq B$.

Inclusions 2/2

Example 1

If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then $A \subseteq B$ and $A \subset B$.

Example 2

If $A = \{1, 2, 4\}$ and $B = \{1, 2, 3\}$, then $A \subseteq B$ does not hold, because 4 belongs to A, but it does not belong to B.

Union of sets 1/2

Union of sets

Let X be a set, Σ be a family of sets from $\mathcal{P}(X)$. **The union of the members from** Σ is the following subset of X:

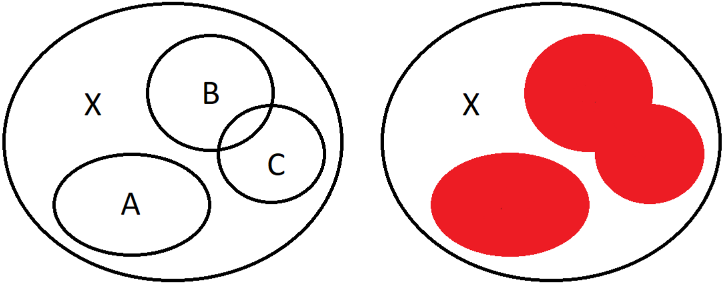
$$\bigcup_{E\in\Sigma} E = \{x \in X \ : \ x \in E \text{ for some } E \in \Sigma\} = \{x \in X \ : \ \exists_{E\in\Sigma} x \in E\}.$$

$\exists \equiv$ there exists.

Union of sets 2/2

Example

If $\Sigma = \{A, B, C\}$, then $\bigcup_{E\in\Sigma} E = A \cup B \cup C$



Intersection of sets 1/3

Intersection of sets

Let X be a set, $\Sigma \neq \emptyset$ be a family of sets from $\mathcal{P}(X)$. **The intersection of the members from** Σ is the following subset of X:

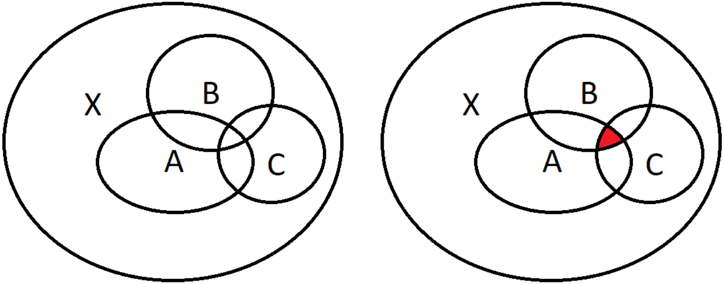
$$\bigcap_{E \in \Sigma} E = \{x \in X \ : \ x \in E \text{ for all } E \in \Sigma\} = \{x \in X \ : \ \forall_{E \in \Sigma} x \in E\}.$$

$\forall \equiv$ for all.

Intersection of sets 2/3

Example 1

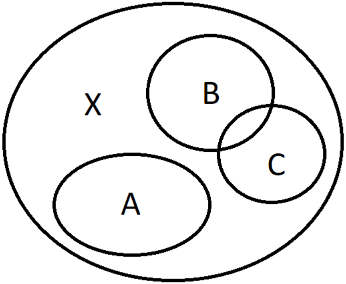
If $\Sigma = \{A, B, C\}$, then $\bigcap_{E\in\Sigma} E = A \cap B \cap C$



Intersection of sets 3/3

Example 2

If $\Sigma = \{A, B, C\}$ as in the picture, then $\bigcap_{E\in\Sigma} E = A \cap B \cap C = \emptyset$.



Union and intersection of indexed family of sets

If $\Sigma = \{E_\alpha : \alpha \in A\}$, then the union and the intersection will be denoted respectively by

$$\bigcup_{\alpha \in A} E_\alpha \text{ and } \bigcap_{\alpha \in A} E_\alpha .$$

Example 1

If $A = \{1, 2, 3\}$, then $\bigcup_{\alpha \in A} E_\alpha = E_1 \cup E_2 \cup E_3$.

Example 2

If $A = \mathbb{N}$, then $\bigcup_{\alpha \in A} E_\alpha = E_1 \cup E_2 \cup E_3 \cup E_4 \cup \dots$.

Disjointness

Definition (Disjointness)

If $A \cap B = \emptyset$, then we say that A and B are **disjoint**.

Example

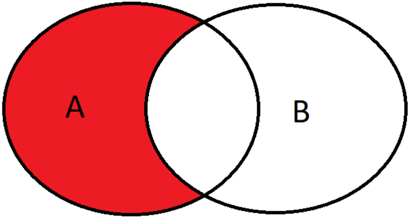
If $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{1, 2, 3\}$, then A and B are disjoint, but A and C are not disjoint.

Difference of sets

Difference of sets

If A, B are two sets, then

$$A \setminus B = \{x \in A \ : \ x \notin B\}.$$



Example 1

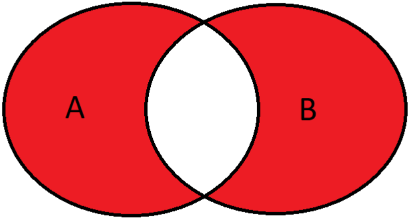
If $A = \{1, 2, 3\}$ and $B = \{3\}$, then $A \setminus B = \{1, 2\}$.

Symmetric difference of sets

Symmetric difference of sets

If A, B are two sets, then

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$



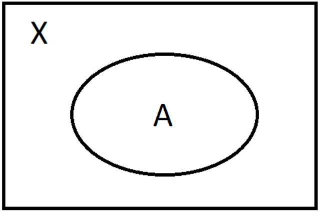
Example

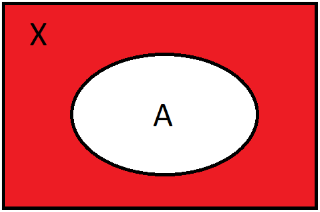
If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then $A \triangle B = \{1, 2, 5, 6\}$.

Complement of sets

Complement of sets

If a set X is given, and $A \subseteq X$, then the complement of A in X is defined by $A^c = X \setminus A$.





de Morgan's laws

de Morgan's laws

$$\left(\bigcup_{\alpha\in A} E_\alpha\right)^c = \bigcap_{\alpha\in A} E_\alpha^c$$

$$\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c$$

Example

We have $(A\cup B)^c = A^c \cap B^c$ and $(A\cap B)^c = A^c \cup B^c$.

Well-ordering principle

Well-ordering principle

If A is a non-empty subset of non-negative integers $\mathbb{N}_0$, then A contains the smallest number.

Example 1

If $A = \{65, 43, 21\}$, then the smallest element is 21.

Example 2

If A is the set of even numbers, then the smallest element is 0.

Induction principle

The principle of induction

If A is a set of non-negative integers such that

A. (Base step): $0 \in A$
B. (Induction step): Whenever A contains a number n, it also contains the number $n+1$.

Then $A = \mathbb{N}_0$.

$$\forall_{A \subseteq \mathbb{N}_0} \left(0 \in A \text{ and } \forall_{k \in \mathbb{N}} (k \in A \Longrightarrow (k+1) \in A) \text{ then } A = \mathbb{N}_0\right)$$

The maximum principle

Subset bounded from above

We say that $A \subseteq \mathbb{N}_0$ is bounded from above if there is $M \in \mathbb{N}_0$ such that $a \leq M$ for all $a \in A$.

$$\exists_{M \in \mathbb{N}_0} \ \ \forall_{a \in A} \ \ a \leq M$$

The maximum principle

A non-empty subset of $\mathbb{N}_0$, which is bounded from above contains the greatest number.

Induction principle - example

Exercise

Prove that for all $n \in \mathbb{N}_0$ we have

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}. \tag{1}$$

Solution. Let A be the set of n for which (1) holds.

$$A = \left\{ n \in \mathbb{N}_0 \ : \ \sum_{k=0}^{n} k = \frac{n(n+1)}{2} \right\}$$

Our goal is to show that $A = \mathbb{N}_0$. We will use **the induction principle**. We have to check the base step and the induction step.

Induction principle - example (base step)

Exercise

Prove that for all $n \in \mathbb{N}_0$ we have

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}. \tag{2}$$

Let us check if $0 \in A$. We have

$$\sum_{k=0}^{0} k = 0 = \frac{0(0+1)}{2},$$

so $0 \in A$.

Induction principle - example (induction step 1/2)

Exercise

Prove that for all $n \in \mathbb{N}_0$ we have

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}.$$

Let us check that whenever $n \in A$, then $n+1 \in A$. If $n \in A$, then

$$\sum_{k=0}^{n} k = 1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2}.$$

Our goal is to prove that $n+1 \in A$, i.e.,

$$\sum_{k=0}^{n+1} k = 1 + 2 + 3 + \ldots + (n-1) + n + (n+1) = \frac{(n+1)(n+2)}{2}.$$

Induction principle - example (induction step 2/2)

Exercise

Prove that for all $n \in \mathbb{N}_0$ we have

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}.$$

We calculate

$$\sum_{k=0}^{n+1} k = 1 + 2 + 3 + \ldots + (n-1) + n + (n+1)$$

$$= 1 + 2 + 3 + \ldots + (n-1) + n + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{n^2+n}{2} + \frac{2n+2}{2} = \frac{(n+1)(n+2)}{2}.$$