Lesson 10

The Limit of a Sequence
The Algebraic and Order Limit Theorems

MATH 311, Section 4, FALL 2022

October 7, 2022

Exercise

Exercise

Two $a, b, \in \mathbb{R}$ are equal iff for every $\varepsilon > 0$ it follows

$$|a - b| < \varepsilon.$$

Proof ($\Leftarrow$). If $a = b$, then $|a - b| = 0 < \varepsilon$ for any $\varepsilon > 0$.

Proof ($\Rightarrow$). Suppose that for any $\varepsilon > 0$ we have $|a - b| < \varepsilon$. If $a = b$, then we are done. Assume that $a \neq b$ and take $\varepsilon_0 = |a - b| > 0$. Taking any $0 < \varepsilon < \varepsilon_0$ one have

$$0 < \varepsilon_0 = |a - b| < \varepsilon < \varepsilon_0,$$

which is impossible. □

Sequences

Definition

A sequence is a function whose domain is $\mathbb{N}$.

Example

Common ways to describe sequences:

(i) $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$,

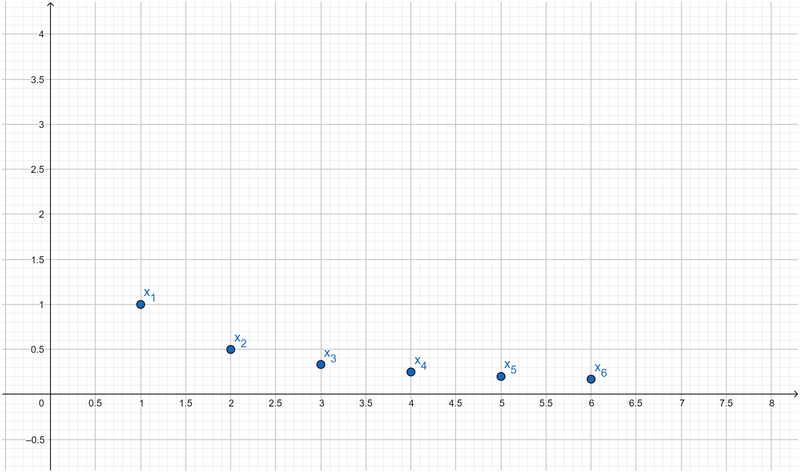
(ii) $\left(\frac{n+1}{n}\right)_{n=1}^{\infty} = \left(\frac{n+1}{n}\right)_{n\in\mathbb{N}} = \left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots\right)$,

(iii) $(x_n)_{n\in\mathbb{N}}$, where $x_n = 2^n$ for each $n \in \mathbb{N}$,

(iv) $(a_n)_{n\in\mathbb{N}}$, where $a_1 = 2$ and $a_{n+1} = \frac{a_n}{2}$.

Graph of a sequence

Consider $x_n = \frac{1}{n}$, then



Asymptotic behaviour of a sequence

Question

Is there a reasonable way how to measure how small the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is asymptotically ($\equiv$ at infinity)?

- We take an arbitrary $\varepsilon > 0$ and since $x_n = \frac{1}{n}$ then by the Archimedian property we always find $N_\varepsilon \in \mathbb{N}$ so that $\frac{1}{N_\varepsilon} < \varepsilon$.
- Moreover, since $x_{n+1} = \frac{1}{n+1} < \frac{1}{n} = x_n$ for every $n \in \mathbb{N}$ thus

$$\frac{1}{n} < \varepsilon \quad \text{for any} \quad n \geq N_\varepsilon. \qquad (*)$$

- Since $\varepsilon > 0$ is arbitrary and (*) holds for all $n \geq N_\varepsilon$ (we will usually say that (*) holds for all but finitely many integers or for all sufficiently large integers).
- One can also think that the sequence $(x_n)_{n\in\mathbb{N}}$ is asymptotically small or small at infinity.

Convergence of a sequence

Convergence of a sequence

A sequence $(x_n)_{n\in\mathbb{N}}$ **converges** to a real number $x \in \mathbb{R}$ if, for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that whenever $n \geq N_\varepsilon$ it follows that

$$|x - x_n| < \varepsilon.$$

To indicate that $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ we will write either

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad \lim a_n = a \quad \text{or} \quad a_n \xrightarrow[n\to\infty]{} a \quad \text{or} \quad a_n \to a.$$

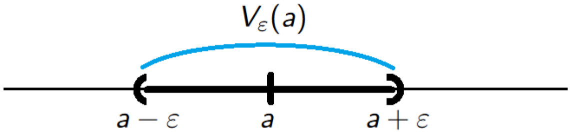
ε-neighbourhood

ε-neighbourhood

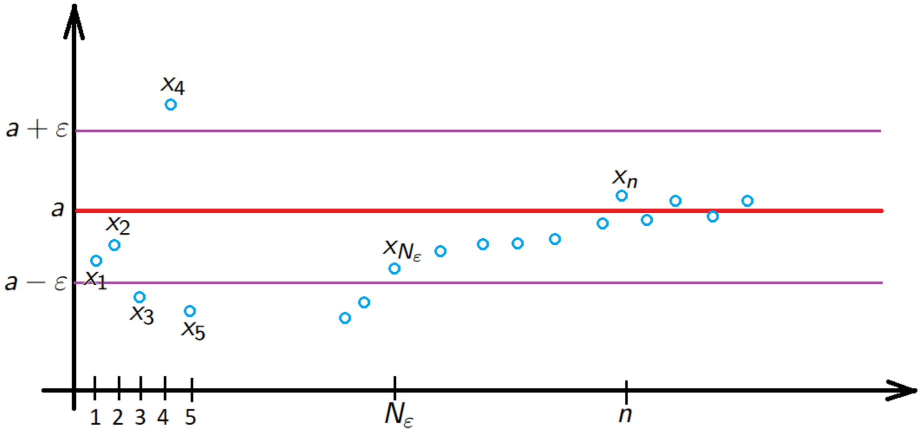
Given $a \in \mathbb{R}$ and $\varepsilon > 0$ the set

$$V_\varepsilon(a) = \{x \in \mathbb{R} \,:\, |x - a| < \varepsilon\}$$

is called the ε-**neighbourhood** or **an open ball** centered at a and radius ε.



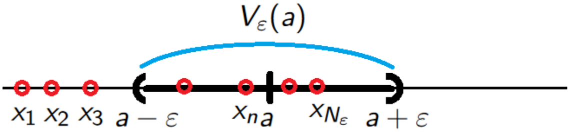
Convergence - illustration



Convergence - topological version

Convergence - topological version

A sequence $(x_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ if, given any ε-neighbourhood $V_\varepsilon(a)$ of a contains all but finitely many terms of $(x_n)_{n\in\mathbb{N}}$.



Convergence of a sequence - exercise 1/2

Exercise

Prove $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

Solution.

1. Let $\varepsilon > 0$ be arbitrary, but fixed.
2. Determine the choice of $N_\varepsilon \in \mathbb{N}$. In our case we take
$$N_\varepsilon = \left\lfloor \frac{1}{\varepsilon^2} \right\rfloor + 1.$$
3. Now show that N_ε actually works. Assume that $n \geq N_\varepsilon$, then
$$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N_\varepsilon}} = \frac{1}{\sqrt{\left\lfloor \frac{1}{\varepsilon^2} \right\rfloor + 1}} < \frac{1}{\sqrt{1/\varepsilon^2}} = \varepsilon.$$

Convergence of a sequence - exercise 2/2

4. With this N_ε, we have $|x_n - x| < \varepsilon$ for all $n \geq N_\varepsilon$. Indeed,

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} < \varepsilon.$$

Hence

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

□

Convergence of a sequence - exercise 1/2

Exercise

Prove $\lim_{n\to\infty} \frac{3n+2}{2n+1} = \frac{3}{2}$.

Solution.

1. Let $\varepsilon > 0$ be arbitrary, but fixed.
2. Determine the choice of $N_\varepsilon \in \mathbb{N}$. In our case we take

$$N_\varepsilon = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1.$$

3. Now show that N_ε actually works. Assume that $n \geq N_\varepsilon$, then

$$\left| \frac{3n+2}{2n+1} - \frac{3}{2} \right| \leq \left| \frac{3n+2}{2n+1} - \frac{3n}{2n+1} \right| + \left| \frac{3n}{2n+1} - \frac{3n}{2n} \right|$$

Convergence of a sequence - exercise 2/2

3. Furthermore, for $n \geq N_\varepsilon$ we have

$$\left|\frac{3n+2}{2n+1} - \frac{3n}{2n+1}\right| = \frac{2}{2n+1} \leq \frac{1}{n} < \frac{\varepsilon}{2}.$$

$$\left|\frac{3n}{2n+1} - \frac{3n}{2n}\right| = \frac{3n(2n+1-2n)}{2n(2n+1)} < \frac{3}{4n} < \frac{\varepsilon}{2}.$$

4. Hence

$$\lim_{n\to\infty} \frac{3n+2}{2n+1} = \frac{3}{2}.$$

□

Convergence of a sequence - exercise 1/2

Exercise

Prove $\lim_{n\to\infty} \frac{n}{n^3+3} = 0$.

Solution.

1. Let $\varepsilon > 0$ be arbitrary, but fixed.
2. Determine the choice of $N_{\varepsilon\in\mathbb{N}}$. In our case we take

$$N_\varepsilon = \left\lfloor \frac{1}{\sqrt{\varepsilon}} \right\rfloor + 1.$$

3. Now show that N_ε actually works. Assume that $n \geq N_\varepsilon$, then

$$\frac{n}{n^3+3} \leq \frac{n}{n^3} = \frac{1}{n^2} < \varepsilon.$$

Convergence of a sequence - exercise 2/2

4. Thus for $n \geq N_\varepsilon$ we have

$$\left| \frac{n}{n^3+3} - 0 \right| = \frac{1}{n^2} < \varepsilon.$$

Hence

$$\lim_{n\to\infty} \frac{n}{n^3+3} = 0.$$

□

Uniqueness of the limit

Uniqueness of the limit

The limit of the sequence, when it exists, must be unique.

Proof. Suppose that

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} x_n = y.$$

We have to prove that $x = y$. Let $\varepsilon > 0$ be arbitrary, then it suffices to show $|x - y| < \varepsilon$. Note that

(*)

$\lim_{n\to\infty} x_n = x \iff$ for every $\varepsilon_1 > 0$ there exists $N^1_{\varepsilon_1} \in \mathbb{N}$ so that

$$n \geq N^1_{\varepsilon_1} \quad \text{implies} \quad |x_n - x| < \varepsilon_1.$$

Proof: uniqueness of the limit

(*)

$\lim_{n\to\infty} x_n = y \iff$ for every $\varepsilon_2 > 0$ there exists $N^2_{\varepsilon_2} \in \mathbb{N}$ so that

$$n \geq N^2_{\varepsilon_2} \quad \text{implies} \quad |x_n - y| < \varepsilon_2.$$

Applying (*) and (*) with $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ we know that there are $N^1_{\varepsilon_1}, N^2_{\varepsilon_2} \in \mathbb{N}$ so that

$$n \geq N^1_{\varepsilon_1} \quad \text{implies} \quad |x_n - x| < \varepsilon_1,$$
$$n \geq N^2_{\varepsilon_2} \quad \text{implies} \quad |x_n - y| < \varepsilon_2.$$

Setting $N_\varepsilon = \max(N^1_{\varepsilon/2}, N^2_{\varepsilon/2})$, taking $n \geq N_\varepsilon$ and using the triangle inequality

$$|x - y| = |(x - x_n) + (x_n - y)| \leq |x_n - x| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Bounded sequence

Bounded sequence

A sequence $(x_n)_{n\in\mathbb{N}}$ is **bounded** if there exists $M > 0$ such that

$$|x_n| \leq M$$

for all $n \in \mathbb{N}$.

Geometrically, this means that the interval $[-M, M]$ contains all terms of the sequence $(x_n)_{n\in\mathbb{N}}$.

Example

- $\left(5 + \frac{1}{n}\right)_{n\in\mathbb{N}}$ is bounded by 6,
- $(n^2)_{n\in\mathbb{N}}$ is not bounded.

Every convergent sequence is bounded

Theorem

Every convergent sequence is bounded.

Proof. Assume that $\lim_{n\to\infty} x_n = x$. This is equivalent to the fact that for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ so that

$$n \geq N_\varepsilon \quad \text{implies} \quad |x_n - x| < \varepsilon. \qquad (*)$$

Applying (*) with $\varepsilon = 1$ we obtain

$$|x_n - x| < 1 \quad \text{for any} \quad n \geq N_1.$$

Thus $|x_n| < 1 + |x|$ for any $n \geq N_1$. Consider

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N_1-1}|, |x| + 1\}$$

we see that $|x_n| \leq M$ for all $n \in \mathbb{N}$ and we are done. □

Algebraic limits theorem

Theorem

Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then

1. (i) $\lim_{n\to\infty}(ca_n) = ac$,
2. (ii) $\lim_{n\to\infty}(a_n + b_n) = a + b$,
3. (iii) $\lim_{n\to\infty} a_n b_n = ab$,
4. (iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided that $b_n, b \neq 0$ for all $n \in \mathbb{N}$.

Proof of (i). If $c = 0$ then there is nothing to do since $ca_n = 0$ for all $n \in \mathbb{N}$, thus $\lim_{n\to\infty} ca_n = 0 = ca$.

- Assume that $c \neq 0$. Let $\varepsilon > 0$ be arbitrary but fixed and note that $\lim_{n\to\infty} a_n = a \iff$ for every $\varepsilon_0 > 0$ there is $N_{\varepsilon_0} \in \mathbb{N}$ such that

$$n \geq N_{\varepsilon_0} \quad \text{implies} \quad |a - a_n| < \varepsilon_0. \qquad (*)$$

Proof of (i)

- Applying (*) with $\frac{\varepsilon}{c}$ in place of ε_0 one gets that

$$|ca_n - ca| = |c||a_n - a| < |c|\frac{\varepsilon}{|c|} = \varepsilon.$$

- Thus we have shown that for any $\varepsilon > 0$ there is $\widetilde{N}_\varepsilon = N_{\varepsilon/|c|} \in \mathbb{N}$ such that if $n \geq \widetilde{N}_\varepsilon$, then

$$|ca_n - ca| < \varepsilon.$$

- Hence $\lim_{n\to\infty} ca_n = ca$. □

Proof of (ii): 1/2

- $\lim_{n\to\infty} a_n = a \iff$ for every $\varepsilon_1 > 0$ there exists $N^1_{\varepsilon_1} \in \mathbb{N}$ so that

$$n \geq N^1_{\varepsilon_1} \quad \text{implies} \quad |a_n - a| < \varepsilon_1. \qquad (*)$$

- $\lim_{n\to\infty} b_n = b \iff$ for every $\varepsilon_2 > 0$ there exists $N^2_{\varepsilon_2} \in \mathbb{N}$ so that

$$n \geq N^2_{\varepsilon_2} \quad \text{implies} \quad |b_n - b| < \varepsilon_2. \qquad (*)$$

- Let $\varepsilon > 0$ be arbitrary but fixed. Applying (*) and (*) with $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$ one obtains

$$\begin{aligned} n \geq N^1_{\varepsilon_1} \quad &\text{implies} \quad |a_n - a| < \varepsilon/2, \\ n \geq N^2_{\varepsilon_2} \quad &\text{implies} \quad |b_n - b| < \varepsilon/2. \end{aligned}$$

Proof of (ii): 2/2

- By the triangle inequality for any $n \geq N_\varepsilon = \max(N^1_{\varepsilon_1}, N^2_{\varepsilon_2})$ we see

$$\begin{aligned} |a_n + b_n - (a+b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

- Since $\varepsilon > 0$ was arbitrary we proved that

$$\lim_{n\to\infty} (a_n + b_n) = a + b.$$

$\square$

Proof of (iii): 1/3

- $\lim_{n\to\infty} a_n = a \iff$ for every $\varepsilon_1 > 0$ there exists $N^1_{\varepsilon_1} \in \mathbb{N}$ so that

$$n \geq N^1_{\varepsilon_1} \quad \text{implies} \quad |a_n - a| < \varepsilon_1. \qquad (*)$$

- $\lim_{n\to\infty} b_n = b \iff$ for every $\varepsilon_2 > 0$ there exists $N^2_{\varepsilon_2} \in \mathbb{N}$ so that

$$n \geq N^2_{\varepsilon_2} \quad \text{implies} \quad |b_n - b| < \varepsilon_2. \qquad (*)$$

- We begin by observing that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\ &\leq |b_n(a_n - a)| + |a(b_n - b)| \\ &\leq |b_n||a_n - a| + |a||b_n - b|. \end{aligned}$$

Proof of (iii): 2/3

- But $|a| \leq |a_n - a| + |a_n|$ thus

$$|a_n b_n - ab| \leq |b_n||a_n - a| + |b_n - b|(|a_n - a| + |a_n|) \\ \leq (|b_n| + |b_n - b|)|a_n - a| + |b_n - b||a_n|.$$

- Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then there are $M_1, M_2 > 0$ such that

$$|a_n| \leq M_1 \quad \text{and} \quad |b_n| \leq M_2 \quad \text{for all} \quad n \in \mathbb{N}.$$

Consequently

$$|a_n b_n - ab| \leq (M_2 + |b_n - b|)|a_n - a| + M_1|b_n - b|.$$

- Let $\varepsilon > 0$ be arbitrary but fixed. We apply (*) with $\varepsilon_1 = \frac{\varepsilon}{2(M_2+1)}$ and (*) with $\varepsilon_2 = \min\left\{\frac{\varepsilon}{2M_1}, 1\right\}$, which implies respectively

Proof of (iii): 3/3

$$n \geq N^1_{\varepsilon/2} \quad \text{implies} \quad |a - a_n| < \frac{\varepsilon}{2(M_2+1)},$$
$$n \geq N^2_{\varepsilon/2} \quad \text{implies} \quad |b - b_n| < \min\left\{\frac{\varepsilon}{2M_1}, 1\right\}.$$

- Thus taking $n \geq N_\varepsilon = \max\left(N^1_{\varepsilon_1}, N^2_{\varepsilon_2}\right)$ we see that

$$\begin{aligned} |a_n b_n - ab| &\leq (M_2 + |b_n - b|)|a_n - a| + M_1|b_n - b| \\ &< \left(M_2 + \min\left\{\frac{\varepsilon}{2M_1}, 1\right\}\right)\frac{\varepsilon}{2(M_2+1)} + M_1 \min\left\{\frac{\varepsilon}{2M_1}, 1\right\} \\ &\leq (M_2+1)\frac{\varepsilon}{2(M_2+1)} + M_1\frac{\varepsilon}{2M_1} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

- Since $\varepsilon > 0$ was arbitrary we proved that

$$\lim_{n\to\infty} a_n b_n = ab.$$

□

Proof of (iv): 1/3

- $\lim_{n\to\infty} a_n = a \iff$ for every $\varepsilon_1 > 0$ there exists $N^1_{\varepsilon_1} \in \mathbb{N}$ so that

$$n \geq N^1_{\varepsilon_1} \quad \text{implies} \quad |a_n - a| < \varepsilon_1. \qquad (*)$$

- $\lim_{n\to\infty} b_n = b \iff$ for every $\varepsilon_2 > 0$ there exists $N^2_{\varepsilon_2} \in \mathbb{N}$ so that

$$n \geq N^2_{\varepsilon_2} \quad \text{implies} \quad |b_n - b| < \varepsilon_2. \qquad (*)$$

- By (iii) it suffices to prove that $\lim_{n\to\infty} b_n = b$ implies

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$$

  whenever $b_n, b \neq 0$ for $n \in \mathbb{N}$.
- Let $\varepsilon > 0$ be arbitrary. Note that

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b_n - b|}{|b_n||b|}.$$

Proof of (iv): 2/3

- Applying (*) with $\varepsilon_2 = \min\left\{\frac{|b|}{2}, \frac{\varepsilon|b|^2}{2}\right\}$ one has

$$n \geq N_{\varepsilon_2} \quad \text{implies} \quad |b_n - b| < \varepsilon_2.$$

- But $\frac{|b|}{2} > |b_n - b| \geq |b| - |b_n|$, hence

$$|b| - |b_n| < \frac{|b|}{2} \quad \text{for all} \quad n \geq N_{\varepsilon_2}.$$

- Consequently $\frac{|b|}{2} < |b_n|$ for all $n \in \mathbb{N}$. This shows that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n||b|} < \frac{2|b_n - b|}{|b|^2} \quad \text{for all} \quad n \geq N_{\varepsilon_2}.$$

Proof of (iv): 3/3

- Furthermore, for $n \geq N_{\varepsilon_2}$ we also know that

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{2|b_n - b|}{|b|^2} < \frac{2\varepsilon_2}{|b|^2} \leq \frac{2\varepsilon|b|^2}{2|b|^2} = \varepsilon.$$

- Thus

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

This completes the proof of the Theorem. □

Order limit theorem

Order limit theorem

Assume that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

1. If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.
2. If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
3. If there is $c \in \mathbb{R}$ so that $c \leq b_n$ for each $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

Proof

Proof of (i). Assume for contradiction that $a < 0$. We know that $\lim_{n\to\infty} a_n = a \iff$ for every $\varepsilon_0 > 0$ there exists $N_{\varepsilon_0} \in \mathbb{N}$ so that

$$n \geq N_{\varepsilon_0} \quad \text{implies} \quad |a_n - a| < \varepsilon_0. \qquad (*)$$

Applying (*) with $\varepsilon_0 = |a|$ one sees

$$|a_n - a| < |a| \quad \text{for all} \quad n \geq N_{\varepsilon_0}.$$

Hence $a_n < 0$ for all $n \geq N_{\varepsilon_0}$ which is impossible since $a_n \geq 0$ for all $n \in \mathbb{N}$. Thus we must have $a \geq 0$. □

Proof of (ii). $\lim_{n\to\infty}(b_n - a_n) = b - a$. But $b_n - a_n \geq 0$ for all $n \in \mathbb{N}$ thus $b - a \geq 0$ by (i) and we are done. □

Proof of (iii). Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbb{N}$ and apply (ii). □