The Squeeze Theorem
The Monotone Convergence Theorem
and other useful theorems

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Convergence of a sequence

Convergence of a sequence

A sequence $(x_n)_{n\in\mathbb{N}}$ **converges** to a real number $x \in \mathbb{R}$ if, for all $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that whenever $n \geq N_\varepsilon$ it follows that

$$|x - x_n| < \varepsilon.$$

Order limit theorem

Assume that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

(i) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iii) If there is $c \in \mathbb{R}$ so that $c \leq b_n$ for each $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

Squeeze Theorem

Squeeze Theorem

If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then $\lim_{n\to\infty} y_n = L$.

Proof. Let $\varepsilon > 0$ be arbitrary, but fixed.

(*)

$\lim_{n\to\infty} x_n = L \iff$ for every $\varepsilon_1 > 0$ there exists $N^1_{\varepsilon_1} \in \mathbb{N}$ so that $n \geq N^1_{\varepsilon_1}$ implies $|x_n - L| < \varepsilon_1$.

(*)

$\lim_{n\to\infty} z_n = L \iff$ for every $\varepsilon_2 > 0$ there exists $N^2_{\varepsilon_2} \in \mathbb{N}$ so that $n \geq N^2_{\varepsilon_2}$ implies $|z_n - L| < \varepsilon_2$.

Proof:

We apply (*) and (*) with $\varepsilon_1 = \varepsilon_2 = \varepsilon$, then for $n \geq N_\varepsilon = \max(N^1_{\varepsilon_1}, N^2_{\varepsilon_2})$ one has

$$(*) \iff L - \varepsilon < x_n < L + \varepsilon,$$

$$(*) \iff L - \varepsilon < z_n < L + \varepsilon.$$

Since $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ we obtain for $n \geq N_\varepsilon$ that

$$L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon.$$

Thus if $n \geq N_\varepsilon$, then

$$|y_n - L| < \varepsilon,$$

which proves that $\lim_{n\to\infty} y_n = L$. □

(MCT) - example

Exercise

Prove that $\lim_{n\to\infty} \sqrt{n^2+1} - n = 0$.

Solution. We will use the **squeeze theorem**. On the one hand,

$$0 \leq \sqrt{n^2+1} - n.$$

On the other hand,

$$\sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{\sqrt{n^2+1} + n} = \frac{n^2+1-n^2}{\sqrt{n^2+1} + n} \leq \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} 0 = 0$, by the squeeze theorem

$$\lim_{n\to\infty} \sqrt{n^2+1} - n = 0.$$

□

Application

Theorem

For every $a \in \mathbb{R}$ there is a sequence of rational numbers $(r_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} r_n = \alpha.$$

Proof. Recall the Dirichlet principle.

Theorem (Dirichlet)

Let α, Q be real numbers, $Q \geq 1$. There exist $a, q \in \mathbb{Z}$ such that $1 \leq q \leq Q$ and $(a, q) = 1$ and

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{qQ} \leq \frac{1}{q^2}.$$

Proof

Applying Dirichlet's theorem with $Q = n \in \mathbb{N}$ one finds for each $n \in \mathbb{N}$ integers $a_n, q_n \in \mathbb{Z}$ such that $1 \leq q_n \leq n$ and $(a_n, q_n) = 1$ and

$$\left|\alpha - \frac{a_n}{q_n}\right| < \frac{1}{q_n n} < \frac{1}{n}.$$

Now let $\varepsilon > 0$ be arbitrary but fixed and let $N_\varepsilon \in \mathbb{N}$ be such that $\frac{1}{N_\varepsilon} < \varepsilon$, then for every $n \geq N_\varepsilon$ one has

$$\frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon.$$

thus taking $r_n = \frac{a_n}{q_n} \in \mathbb{Q}$ and $n \geq N_\varepsilon$ one gets

$$|\alpha - r_n| < \frac{1}{n} < \varepsilon.$$

Hence $\lim_{n\to\infty} r_n = \alpha$. □

Increasing and decreasing sequences

Increasing and decreasing sequences

A sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is

- **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$;
- **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Monotone sequence

A sequence is **monotone** if it is either increasing or decreasing.

Example

- $\left(3 + \frac{1}{n}\right)_{n\in\mathbb{N}}$ is decreasing, so it is monotone.
- $\left(n^3\right)_{n\in\mathbb{N}}$ is increasing, so it is monotone.
- $((-1)^n)_{n\in\mathbb{N}}$ is neither increasing nor decreasing, so it is not monotone.

Monotone convergence theorem

Monotone convergence theorem (MCT)

If a sequence is monotone and bounded then it converges.

Proof. Assume that $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded. Consider the set

$$E = \{x_n \colon n \in \mathbb{N}\} \subseteq \mathbb{R},$$

which is nonempty and bounded. Let $x = \sup E \in \mathbb{R}$, which exists by the axiom of completeness (AoC). We will show that $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$ and note that there exists $N_\varepsilon \in \mathbb{N}$ so that

$$x - \varepsilon < x_{N_\varepsilon} \leq x.$$

But $(x_n)_{n\in\mathbb{N}}$ is increasing thus for any $n \geq N_\varepsilon$ one has

$$x - \varepsilon < x_{N_\varepsilon} \leq x_n \leq x < x + \varepsilon.$$

Hence $|x_n - x| < \varepsilon$ for all $n \geq N_\varepsilon$, which shows that $\lim_{n\to\infty} x_n = x$. □

Binomial theorem

Binomial theorem

For every $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ one has

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, for all $n \in \mathbb{N}$ and $0! = 1$.

Example for $n = 3$:

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

Example for $n = 5$:

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Theorem

Theorem

1. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
2. If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
4. If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
5. If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof of (a): Take $\varepsilon > 0$ be arbitrary, but fixed. Then

$$n > \left(\frac{1}{\varepsilon}\right)^{1/p},$$

which is possible by the Archimedian property.

Proof of (b):

Proof of (b): If $p > 1$ set $x_n = \sqrt[n]{p} - 1$, then $x_n > 0$ and by **Bernoulli's inequality**

$$1 + nx_n \leq (1 + x_n)^n = p,$$

so that

$$0 < x_n \leq \frac{p-1}{n}.$$

But

$$\lim_{n\to\infty} \frac{p-1}{n} = 0,$$

thus by the squeeze theorem we conclude

$$\lim_{n\to\infty} x_n = 0$$

as desired.

Proof of (c):

Proof of (c): Set $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and by the **binomial theorem**

$$n = (1 + x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \leq x_n \leq \left(\frac{2}{n-1}\right)^{1/2} \quad \text{for} \quad n \geq 2.$$

But

$$\lim_{n\to\infty} \left(\frac{2}{n-1}\right)^{1/2} = 0.$$

Thus by the squeeze theorem

$$\lim_{n\to\infty} x_n = 0$$

as desired. □

Proof of (d) and (e):

Proof of (d): Let $k \in \mathbb{N}$ so that $k > \alpha$. For $n > 2k$ by the **binomial theorem**

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!},$$

since $n \geq \frac{n}{2}, n-1 \geq \frac{n}{2}, \dots, n-k+1 \geq \frac{n}{2}$. Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} \quad \text{for} \quad n > 2k.$$

Since $\alpha - k < 0$ thus $\lim_{n\to\infty} n^{\alpha-k} = 0$ by (a) and by the squeeze theorem $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$. □

Proof of (e): Take $\alpha = 0$ in (d) and observe that if $0 < x < 1$ then the sequence $x_n = x^n$ is decreasing and bounded. Thus $\lim_{n\to\infty} x_n = 0$. □

Proposition

Proposition

If $a > 0$ and $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} a^{x_n} = a^{x_0}$.

Proof. It suffices to prove that $\lim_{n\to\infty} a^{x_n} = 1$ if $\lim_{n\to\infty} x_n = 0$. Assume $a > 1$. By the previous theorem we know that

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} a^{-1/n} = 1.$$

Thus for any $\varepsilon > 0$ there is $M_\varepsilon \in \mathbb{N}$ such that for any $m \geq M_\varepsilon$

$$1 - \varepsilon < a^{-1/m} < a^{1/m} < 1 + \varepsilon.$$

Now since $\lim_{n\to\infty} x_n = 0$ we find $N_{m,\varepsilon} \in \mathbb{N}$ so that for $n \geq N_{\varepsilon,m}$

$$|x_n| < \frac{1}{m} \iff -\frac{1}{m} < x_n < \frac{1}{m}.$$

Proof of Proposition

Thus

$$1 - \varepsilon < a^{-1/m} < a^{x_n} < a^{1/m} < 1 + \varepsilon$$

which proves $|a^{x_n} - 1| < \varepsilon$ for any $n \geq N_{m,\varepsilon}$ proving that

$$\lim_{n\to\infty} a^{x_n} = 1.$$

If $0 < a < 1$ we note that

$$\lim_{n\to\infty} a^{x_n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{a}\right)^{x_n}}$$

and this completes the proof of the proposition. □

Weighted arithmetic and geometric means

Theorem

For all positive real numbers $a_1, a_2, \ldots, a_n$ and all positive weights $q_1, q_2, \ldots, q_n$ satisfying the following convexity condition

$$q_1 + \ldots + q_n = 1,$$

we have

$$a_1^{q_1} \cdot \ldots \cdot a_n^{q_n} \leq q_1 a_1 + \ldots + q_n a_n.$$

If $q_1 = q_2 = \ldots = q_n = \frac{1}{n}$, then we have

$$a_1^{q_1} \cdot \ldots \cdot a_n^{q_n} = (a_1 \cdot \ldots \cdot a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n} = q_1 a_1 + \ldots + q_n a_n,$$

which recovers the inequality between geometric and arithmetic means.

Proof: $1/2$

Proof: We first assume

$$q_1, \ldots, q_n \in \mathbb{Q} \quad \text{and} \quad q_1, \ldots, q_n > 0.$$

We can assume that $q_i = \frac{k_i}{m}$ for $1 \leq i \leq n$ and

$$k_1 + \ldots + k_n = m.$$

Invoking the inequality between geometric and arithmetic means we obtain

$$\begin{aligned} \sum_{i=1}^{n} q_i a_i &= k_1 \frac{a_1}{m} + \ldots + k_n \frac{a_n}{m} \\ &\geq m \left(\left(\frac{a_1}{m} \right)^{k_1} \cdot \ldots \cdot \left(\frac{a_n}{m} \right)^{k_n} \right)^{1/m} \\ &= a_1^{k_1/m} \cdot \ldots \cdot a_n^{k_n/m} \\ &= a_1^{q_1} \cdot \ldots \cdot a_n^{q_n}. \end{aligned}$$

Proof: 2/2

If now all weights $q_1, \ldots, q_n > 0$ are real numbers, then for any $1 \leq i \leq n$, we choose a sequence of positive rationals $(q_{i,k})_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} q_{i,k} = q_i$$

and so that

$$\sum_{i=1}^{n} q_{i,k} = 1 \quad \text{for all} \quad k \in \mathbb{N}.$$

Then by the previous part

$$a_1^{q_{i,k}} \cdot \ldots \cdot a_n^{q_{n,k}} \leq q_{1,k}a_1 + \ldots + q_{n,k}a_n.$$

Passing with $k \to \infty$ we conclude that

$$a_1^{q_1} \cdot \ldots \cdot a_n^{q_n} \leq q_1 a_1 + \ldots + q_n a_n. \quad \square$$

Hölder's inequality

Hölder's inequality

Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ one has

$$\sum_{j=1}^{n} |x_j y_j| \le \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |y_j|^q\Big)^{1/q}.$$

Proof. By the previous theorem for any $a_1, b_1 > 0$ we have

$$a_1^{\frac{1}{p}} b_1^{\frac{1}{q}} \le \frac{1}{p} a_1 + \frac{1}{q} b_1,$$

which for $a_1 = a^p$ and $b_1 = b^q$ yields

(*)

$$ab \le \frac{1}{p} a^p + \frac{1}{q} b^q.$$

Proof of Hölder's inequality 1/2

Let

$$a_j := \frac{|x_j|}{\left(\sum_{j=1}^n |x_j|^p\right)^{1/p}}, \quad b_j := \frac{|y_j|}{\left(\sum_{j=1}^n |y_j|^q\right)^{1/q}}$$

Applying (*) to each $1 \leq j \leq n$ one gets

$$\sum_{j=1}^n a_j b_j \leq \sum_{j=1}^n \left(\frac{1}{p}a_j^p + \frac{1}{q}b_j^q\right)$$

$$= \sum_{j=1}^n \left(\frac{|x_j|^p}{p\left(\sum_{j=1}^n |x_j|^p\right)} + \frac{|y_j^q|}{q\left(\sum_{j=1}^n |y_j|^q\right)}\right)$$

$$= \frac{1}{p}\frac{\sum_{j=1}^n |x_j|^p}{\sum_{j=1}^n |x_j|^p} + \frac{1}{q}\frac{\sum_{j=1}^n |y_j|^q}{\sum_{j=1}^n |y_j|^q} = \frac{1}{p} + \frac{1}{q} = 1.$$

Proof of Hölder's inequality 2/2

Thus we have proved

$$\sum_{j=1}^{n} a_j b_j \leq 1,$$

which is equivalent to

$$\sum_{j=1}^{n} |x_j y_j| \leq \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{1/p} \Big(\sum_{j=1}^{n} |y_j|^q \Big)^{1/q}$$

and the proof of Hölder's inequality is completed. □