Lesson 12

Euler's numbers
Subsequences
A First Glance at Infinite Series

MATH 311, Section 4, FALL 2022

October 18, 2022

Divergence of a sequence

Divergence of a sequence

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ **diverges to** $+\infty$ and write

$$\lim_{n\to\infty} a_n = +\infty$$

iff for any $M > 0$ there exists $N_M \in \mathbb{N}$ such that for all $n \geq N_M$

$$a_n > M.$$

- We have a similar definition for $\lim_{n\to\infty} a_n = -\infty$.

Example

$(n^2)_{n\in\mathbb{N}}$ diverges to $+\infty$, whereas $(\sqrt{n} - n)_{n\in\mathbb{N}}$ diverges to $-\infty$.

Euler's sequences: 1/4

Consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ defined by

$$a_n = \left(1+\frac{1}{n}\right)^n, \qquad b_n = \left(1+\frac{1}{n}\right)^{n+1} \quad \text{for all} \quad n \in \mathbb{N}$$

We have the following properties.

1. Observe that $a_n < b_n$ for all $n \in \mathbb{N}$. Indeed,

$$a_n = \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n}\right)^{n+1} = b_n,$$

since $1 < 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$.

Euler's sequences: 2/4

2. The sequence $(a_n)_{n\in\mathbb{N}}$ is strictly increasing, i.e.

$$a_n < a_{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. By the geometric-arithmetic mean inequality $G_{n+1} < A_{n+1}$ (which is strict unless $x_1 = x_2 = \ldots = x_{n+1}$) with

$$x_1 = 1 \quad \text{and} \quad x_2 = x_3 = \ldots = x_{n+1} = 1 + \frac{1}{n},$$

we obtain

$$G_{n+1} = \left(\left(1+\frac{1}{n}\right)^n\right)^{1/(n+1)} < \frac{1+n\left(1+\frac{1}{n}\right)}{n+1} = 1 + \frac{1}{n+1} = A_{n+1}.$$

Thus

$$a_n = \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} = a_{n+1}.$$

□

Euler's sequences: 3/4

3. The sequence $(b_n)_{n\in\mathbb{N}}$ is strictly increasing, i.e.

$$b_{n+1} < b_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. By the harmonic-geometric mean inequality $H_{n+1} < G_{n+1}$ (which is strict unless $x_1 = x_2 = \ldots = x_{n+1}$) with

$$x_1 = 1 \quad \text{and} \quad x_2 = x_3 = \ldots = x_{n+1} = 1 + \frac{1}{n-1} = \frac{n}{n-1}.$$

Then

$$H_{n+1} = \frac{n+1}{1 + n\frac{n-1}{n}} < \left(1 + \frac{1}{n-1}\right)^{n/(n+1)} = G_{n+1},$$

thus

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1} < \left(1 + \frac{1}{n-1}\right)^{n} = b_{n-1}. \quad \square$$

Euler's sequences: 4/4

Collecting (1),(2),(3) we have

$$2 = a_1 < a_n < b_n < b_1 = 4 \quad \text{for all} \quad n \geq 2.$$

By the (MCT) the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) a_n = \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\right) \left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} a_n.$$

Euler number

The limit of the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is called **the Euler number**

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1} = e \simeq 2,718\ldots.$$

Subsequences

Definition

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and $n_1 < n_2 < \ldots < n_k < \ldots$ be an increasing sequence of positive integers. Then the sequence

$$(a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots)$$

is called **a subsequence** of $(a_n)_{n\in\mathbb{N}}$ and is denoted by $(a_{n_k})_{k\in\mathbb{N}}$.

Example

Let $(a_n)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots, \right)$, then $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$ and $\left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots\right)$ are subsequences of $(a_n)_{n\in\mathbb{N}}$. The sequences

$$\left(\frac{1}{10}, \frac{1}{2}, \frac{1}{100}, \ldots\right) \quad \text{and} \quad (1, 1, \ldots) \quad \text{are NOT!}.$$

Limit of a subsequence

Theorem

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Assume $\lim_{n\to\infty} a_n = a$ and let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence. Given $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ so that

$$n \geq N_\varepsilon \qquad \text{implies} \qquad |a_n - a| < \varepsilon.$$

Because $n_k \geq k$ for all $k \in \mathbb{N}$, the same N_ε will suffice for the subsequence, that is

$$|a_{n_k} - a| < \varepsilon \qquad \text{whenever} \qquad k \geq N_\varepsilon.$$

□

Euler's number - fact

Fact

If $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, then

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e.$$

In particular, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for any $x \in \mathbb{N}$.

Proof. Let $\lim_{n\to\infty} a_n = +\infty$ and consider $b_n = \lfloor a_n \rfloor$. Then $b_n \leq a_n < b_n + 1$, hence

$$\left(1+\frac{1}{b_n+1}\right)^{b_n} < \left(1+\frac{1}{a_n}\right)^{a_n} < \left(1+\frac{1}{b_n}\right)^{b_n+1}.$$

Proof: 1/4

By the squeeze theorem it suffices to prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{b_n+1}\right)^{b_n}=\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n+1}=e$$

or even

$$\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n}=e.$$

- If $(b_n)_{n\in\mathbb{N}}$ were increasing then as a subsequence of $(n)_{n\in\mathbb{N}}$ we could conclude $\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n}=e$, since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$.
- But we only know that $\lim_{n\to\infty} b_n=+\infty$. It does not mean that $(b_n)_{n\in\mathbb{N}}$ is increasing.

Proof: 2/4

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ we can find $\widetilde{N}_\varepsilon \in \mathbb{N}$ so that $n \geq \widetilde{N}_\varepsilon$ implies

$$\left|\left(1+\frac{1}{n}\right)^n - e\right| < \varepsilon.$$

But $\lim_{n\to\infty} b_n = +\infty$ thus we can find $N_\varepsilon \in \mathbb{N}$ so that $n \geq N_\varepsilon$ implies $b_n \geq \widetilde{N}_\varepsilon$. In particular, we conclude that

$$\left|\left(1+\frac{1}{b_n}\right)^{b_n} - e\right| < \varepsilon$$

for all $n \geq N_\varepsilon$ and thus

$$\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n} = e.$$

Consequently, $\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n}$ as $\lim_{n\to\infty} a_n = +\infty$.

Proof: 3/4

Moreover,

$$\lim_{n\to\infty}\left(1-\frac{1}{a_n}\right)^{a_n}=e^{-1},$$

because

$$\lim_{n\to\infty}\left(1-\frac{1}{a_n}\right)^{a_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{a_n-1}\right)^{a_n}}=\frac{1}{e}.$$

this implies

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n}=e \quad \text{if} \quad \lim_{n\to\infty}a_n=-\infty.$$

Proof: 4/4

For the second part we take

$$a_n = \frac{n}{x},$$

then either

$$\lim_{n\to\infty} a_n = +\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty.$$

Hence

$$\lim_{n\to\infty} \left[\left(1 + \frac{1}{a_n} \right)^{a_n} \right]^x = e^x.$$

- Here we have used the following simple fact: if $\lim_{n\to\infty} a_n = a$, then for any $\alpha \in \mathbb{R}$ we have

$$\lim_{n\to\infty} a_n^\alpha = a^\alpha.$$

Prove it!

Limit of a subsequence - example

Exercise

Find $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{4n}$.

Solution. Since $(2n)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}=e.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{4n}&=\left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)\\&=e\cdot e=e^2.\end{aligned}$$

□

Limit of a subsequence - example

Exercise

Find $\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{4n^2+1}$.

Solution. Since $(n^2+1)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1}=e.$$

Therefore,

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{4n^2+1}$$
$$=\left(\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1}\right)^4\left(\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{-3}\right)=e^4. \quad \square$$

Series

Convergence of a series

Let $(b_n)_{n\in\mathbb{N}}$ be a sequence. **An infinite series** is a formal expression of the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots$$

We define the corresponding sequence of **partial sums** $(s_n)_{n\in\mathbb{N}}$ by

$$s_m = \sum_{n=1}^{m} b_n = b_1 + b_2 + \ldots + b_m.$$

We say that $\sum_{n=1}^{\infty} b_n$ **converges to** B if

$$\lim_{n\to\infty} s_n = B.$$

Example

Exercise

If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, the series diverges.

Solution. If $x < 1$, then

$$s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

and the result follows if we let $n \to \infty$.

For $x \geq 1$ note that

$$\underbrace{1 + 1 + \ldots + 1}_{n} \leq s_n.$$

We have $\lim_{n\to\infty} n = +\infty$, thus $\lim_{n\to\infty} s_n = +\infty$.

Example

Exercise

$$\sum_{n=1}^{\infty} \frac{1}{k^2} < \infty.$$

Solution. Because the terms in the sum are all positive the sequence

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} \quad \text{is increasing.}$$

We now show that $(s_n)_{n \in \mathbb{N}}$ is bounded.

- Then the (MCT) will prove that the series converges.

Solution

To prove boundedness of $(s_n)_{n\in\mathbb{N}}$ we note that

$$
\begin{aligned}
s_n &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \ldots + \frac{1}{n\cdot n} \\
&< 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(n-1)n} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 2 - \frac{1}{n} < 2.
\end{aligned}
$$

Thus by the (MCT) the limit $\lim_{n\to\infty} s_n$ exists. □

- One can also prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. This is also Euler's result.

An example of a diverging series

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Solution. Note that

$$\begin{aligned}
1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) + \left(\frac{1}{17} + \ldots\right. \\
\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16} + 16 \cdot \frac{1}{32} + \ldots \\
= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = 1 + \lim_{n\to\infty} \frac{n}{2} = \infty.
\end{aligned}$$

Thus $s_n = \sum_{k=1}^{n} \frac{1}{k} \xrightarrow[n\to\infty]{} \infty$. □

Cauchy Condensation Test

Cauchy Condensation Test

Suppose that $(b_n)_{n\in\mathbb{N}}$ is decreasing and $b_n \geq 0$ for all $n \in \mathbb{N}$. Then the series

$$\sum_{n=1}^{\infty} b_n < \infty \quad \text{converges}$$

iff the series

$$\sum_{n=1}^{\infty} 2^n b_{2^n} < \infty \quad \text{converges.}$$

Proof. Let

$$s_n = b_1 + b_2 + \ldots + b_n,$$

$$t_k = b_1 + 2b_2 + \ldots + 2^k b_{2^k}.$$

Proof: 1/2

For $n < 2^k$ one has

$$s_n \le b_1 + \overbrace{b_2 + b_3}^{2} + \ldots + \overbrace{b_{2^k} + \ldots + b_{2^{k+1}-1}}^{2^k}$$
$$\le b_1 + 2b_2 + \ldots + 2^k b_{2^k} = t_k.$$

(*)

so that $s_n \le t_k$ for $n < 2^k$.

Proof: 2/2

If $n > 2^k$ one has

$$s_n \geq b_1 + b_2 + (b_3 + b_4) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k})$$
$$\geq \frac{1}{2}b_1 + b_2 + 2b_4 + \ldots + 2^{k-1}b_{2^k} = \frac{1}{2}t_k.$$

(**)

Thus $2s_n \geq t_k$ for $n > 2^k$.

- By (*) and (**) the sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ are either both bounded or both unbounded. □

Corollary

Corollary

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \text{iff} \quad p > 1$$

Proof. The sequence $b_n = \frac{1}{n^p}$ is decreasing and $b_n \geq 0$ for all $n \in \mathbb{N}$. By the Cauchy condensation test we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \frac{2^n}{2^{pn}} < \infty.$$

But the latter converges provided that

$$\sum_{n=1}^{\infty} \frac{2^n}{2^{pn}} = \sum_{n=1}^{\infty} 2^{(1-p)n} = \frac{1}{1 - \frac{1}{2^{p-1}}} < \infty \quad \Longleftrightarrow \quad p > 1. \quad \square$$

Theorem

Theorem

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Proof. Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. Then

1. $s_n < s_{n+1}$ for all $n \in \mathbb{N}$,
2. $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} < 2 + \sum_{k=2} \frac{1}{2^{k-1}} < 3.$

Thus the limit $\lim_{n\to\infty} s_n$ exists.

Let $t_n = \left(1 + \frac{1}{n}\right)^n$, then $\lim_{n\to\infty} t_n = e$. By the binomial theorem

$$t_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}.$$

Proof: 1/2

Then

$$\begin{aligned} t_n &= \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{1}{n^k} \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots \\ &+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot \ldots \cdot \left(1-\frac{n-1}{n}\right) = \sum_{k=0}^{n} \frac{1}{k!} = s_n. \end{aligned}$$

Thus

$$e = \lim_{n\to\infty} t_n \leq \lim_{n\to\infty} s_n.$$

Proof: 2/2

Next if $n \geq m$

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

Let $n \to \infty$ keeping m fixed, we get

$$e = \lim_{n\to\infty} t_n \geq \sum_{k=0}^{m} \frac{1}{k!}.$$

Letting $m \to \infty$ we see $\lim_{m\to\infty} s_m \leq e$.

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} = e.$$

This completes the proof of the theorem. □

Remark

We have $s_n = \sum_{k=0}^{n} \frac{1}{k!} < e$ for all $n \in \mathbb{N}$. Indeed

$$
\begin{aligned}
e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \\
&= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots\right) \\
&< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) \\
&\leq \frac{1}{(n+1)!}\frac{1}{1 - \frac{1}{n+1}} = \frac{1}{(n+1)!}\frac{n+1}{n} = \frac{1}{n!n}.
\end{aligned}
$$

Hence we conclude

The error estimate (*)

$$0 < e - s_n < \frac{1}{n!n}.$$

Euler's number e is irrational

Theorem

The Euler number e is irrational.

Proof. Suppose e is rational. Then $e = \frac{p}{q}$ where $p, q \in \mathbb{N}$. By (*) we have

$$0 < q!(e - s_q) < \frac{1}{q}.$$

By our assumption

$$q!e \in \mathbb{N} \quad \text{is an integer.}$$

Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\right) \in \mathbb{N},$$

we see $q!(e - s_q) \in \mathbb{N}$, but if $q > 1$ and this is impossible since

$$0 < q!(e - s_q) < 1/q < 1.$$

Hence e must be irrational. □