Lesson 13

Toeplitz theorem and applications, Exponential and logarithm function, Bolzano–Weierstrass theorem

MATH 311, Section 4, FALL 2022

October 21, 2022

Toeplitz theorem

Toeplitz theorem

Let $\{c_{n,k} \ : \ 1 \leq k \leq n, n \geq 1\}$ be an array or real numbers such that

1. $\lim_{n\to\infty} c_{n,k} = 0$ for $k \in \mathbb{N}$.
2. $\lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k} = 1$.
3. There is $C > 0$ such that

$$\sup_{n\in\mathbb{N}} \sum_{k=1}^{n} |c_{n,k}| \leq C.$$

Then for any sequence $(a_n)_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} a_n = a$ its transformated sequence

$$b_n = \sum_{k=1}^{n} c_{k,n} a_k$$

also converges and $\lim_{n\to\infty} b_n = a$.

Toeplitz theorem - example

$$\begin{bmatrix} 1 & & & & & & & & \cdots \\ \frac{1}{2} & \frac{1}{2} & & & & & & & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & & & & & & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & & & & & \cdots \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & & & & \cdots \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & & & \cdots \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & & \cdots \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

Proof: 1/2

If $a_n = a$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} b_n = a \lim_{n\to\infty} c_{n,k} = a.$$

Thus, it suffices to consider the case when $\lim_{n\to\infty} a_n = 0$.

Then, for any $m > 1$ and $n \geq m$:

(*)

$$|b_n - 0| = \left|\sum_{k=1}^{n} c_{n,k} a_k\right| \leq \sum_{k=1}^{m-1} |c_{n,k}||a_k| + \sum_{k=m}^{n} |c_{n,k}||a_k|.$$

Since $\lim_{n\to\infty} a_n = 0$ thus for any $\varepsilon > 0$ there is $N_\varepsilon^1 \in \mathbb{N}$ so that

$$n \geq N_\varepsilon^1 \quad \text{implies} \quad |a_n| < \frac{\varepsilon}{2C}.$$

Proof: 2/2

Of course $|a_n| \leq M$ for all $n \in \mathbb{N}$ and some $M > 0$. If follows from (i) that there is $N_\varepsilon^2 \in \mathbb{N}$ so that $n \geq N_\varepsilon^2$ implies

$$\sum_{k=1}^{N_\varepsilon^1 - 1} |c_{n,k}| < \frac{\varepsilon}{2M}.$$

Applying (*) with $m = N_\varepsilon^1$ we obtain

$$|b_n| \leq M \sum_{k=1}^{N_\varepsilon^1 - 1} |c_{n,k}| + \frac{\varepsilon}{2C} \sum_{k=N_\varepsilon^1}^{n} |c_{n,k}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq \max(N_\varepsilon^1, N_\varepsilon^2)$. Thus we conclude

$$\lim_{n \to \infty} b_n = 0.$$

□

Proposition

Proposition

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences such that

1. $b_n > 0$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_k = +\infty$,
2. $\lim_{n\to\infty} \frac{a_n}{b_n} = g$.

then

$$\lim_{n\to\infty} \frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n} = g.$$

Proof. We apply Toeplitz theorem to the sequence $(a_n/b_n)_{n\in\mathbb{N}}$ with

$$c_{n,k} = \frac{b_k}{b_1 + \ldots + b_n}.$$

Proof

Then for (i) we note

$$\lim_{n\to\infty} c_{n,k} = \lim_{n\to\infty} \frac{b_k}{b_1 + \ldots + b_n} = b_k \lim_{n\to\infty} \frac{1}{b_1 + \ldots + b_n} = b_k \cdot 0 = 0.$$

For (ii) we have

$$\sum_{k=1}^{n} c_{n,k} = \sum_{k=1}^{n} \frac{b_k}{b_1 + \ldots + b_n} = \frac{b_1 + \ldots + b_n}{b_1 + \ldots + b_n} = 1.$$

Since $b_k > 0$ (iii) follows from (ii). Thus by Toeplitz theorem we conclude

$$g = \lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k} \frac{a_k}{b_k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k b_k}{(b_1 + \ldots + b_n) b_k} = \lim_{n\to\infty} \frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n}.$$

□

Proposition

Proposition

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences such that

1. $b_n > 0$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} \sum_{k=1}^{n} b_k = +\infty$.
2. $\lim_{n\to\infty} a_n = a$.

Then

$$\lim_{n\to\infty} \frac{a_1 b_1 + \ldots + a_n b_n}{b_1 + \ldots + b_n} = a.$$

Proof. We apply Toeplitz theorem with the sequence $(a_n)_{n\in\mathbb{N}}$ and

$$c_{n,k} = \frac{b_k}{b_1 + \ldots + b_n}.$$

Then

$$a = \lim_{n\to\infty} \sum_{k=1}^{n} c_{n,k} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k b_k}{b_1 + \ldots + b_n} = \lim_{n\to\infty} \frac{a_1 b_1 + \ldots + a_n b_n}{b_1 + \ldots + b_n}. \qquad \square$$

Stolz theorem

Stolz theorem

Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be two sequences so that

(i) $(y_n)_{n\in\mathbb{N}}$ strictly increases to $+\infty$, i.e. $y_n < y_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = +\infty$.

(ii) Also we have

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = g.$$

then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = g.$$

Proof. Apply the previous proposition with

$$a_n = \frac{x_n - x_{n-1}}{y_n - y_{n-1}}, \quad \text{and} \quad b_n = y_n - y_{n-1}.$$

□

Euler's number

We know that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

for any $x \in \mathbb{R}$.

Also

$$\sum_{n=0}^{\infty}\frac{1}{n!} = e.$$

Theorem

Let $x \in \mathbb{R}$, then

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x.$$

Proof: 1/2

Proof. Let $S_n = \sum_{k=0}^{n} \frac{x^k}{k!}$, then by the binomial theorem we may write

$$\left|S_n - \left(1+\frac{x}{n}\right)^n\right| = \left|\sum_{k=2}^{n}\left(1-\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\right)\frac{x^k}{k!}\right|$$
$$\le \sum_{k=2}^{n}\left(1-\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right)\right)\frac{|x|^k}{k!}.$$

Let us also note that

$$\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right) \ge 1-\sum_{j=1}^{k-1}\frac{j}{n} = 1-\frac{k(k-1)}{n}$$

for $2 \le k \le n$.

Proof: 2/2

Thus

$$\left|S_n - \left(1 + \frac{x}{n}\right)^n\right| \leq \sum_{k=2}^{n} \frac{k(k-1)}{2n} \frac{|x|^k}{k!} = \frac{1}{2n} \sum_{k=2}^{n} \frac{|x|^k}{(k-2)!}.$$

Using the Stolz theorem

$$\lim_{n\to\infty} \frac{1}{2n} \sum_{k=2}^{n} \frac{|x|^k}{(k-2)!} = \lim_{n\to\infty} \frac{1}{2} \frac{|x|^n}{(n-2)!} = 0.$$

Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

as desired. □

Exponential function

Exponential function

The function $E : \mathbb{R} \to (0, \infty)$ defined by $E(x) = e^x$ is called **the exponential function**.

Properties of exponential function

1. For all $x, y \in \mathbb{R}$ one has
$$e^{x+y} = e^x e^y.$$
2. If $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} e^{a_n} = e^a$.
3. E is one-to-one and onto. Thus the inverse for E exists.

Natural logarithm

Natural logarithm

The inverse of E exists. It will be denoted by $E^{-1} : (0, \infty) \to \mathbb{R}$,

$$E^{-1}(x) = \ln(x) = \log(x)$$

and it is called **the natural logarithm**.

Simple properties of natural logarithm

1. $\log(x)$ is increasing.
2. For $x, y \in (0, \infty)$ we have

$$\log(xy) = \log(x) + \log(y).$$

3. We also have $x^\alpha = e^{\alpha \log(x)}$ for all $\alpha \in \mathbb{R}$.

Proposition

Proposition

For $x > 0$ we have

$$\frac{x}{x+2} < \log(x+1) < x.$$

Proof. We prove that for $0 < x < m$ with $m \in \mathbb{N}$, we have

$$\left(1+\frac{x}{n}\right)^n < e^x < \left(1+\frac{x}{n}\right)^{n+m}.$$

thus

$$n\log\left(1+\frac{x}{n}\right) < x < (n+m)\log\left(1+\frac{x}{n}\right).$$

Hence

$$\frac{x}{n+m} < \log\left(1+\frac{x}{n}\right) < \frac{x}{n} \quad \text{if} \quad m > x.$$

Proof:

Taking $n = 1$ we obtain

$$\log(1+x) < x \quad \text{for all} \quad x > 0.$$

Now set $m = \lfloor x \rfloor + 1 > x$, then

$$\log\left(1 + \frac{x}{n}\right) > \frac{\frac{x}{n}}{2 + \frac{x}{n}}.$$

Thus for $n = 1$ we obtain

$$\log(1+x) > \frac{x}{2+x}.$$

□

Remark

In fact, for every $x > 0$ the following inequality holds

$$\frac{x}{x+1} < \log(x+1) < x.$$

Euler–Mascheroni constant

Divergence of harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Theorem

The sequences

$$a_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log(n) \quad \text{and} \quad b_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n)$$

are increasing and decreasing respectively and bounded, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \gamma.$$

where γ is known as **the Euler (or Euler–Mascheroni) constant**.

Proof: 1/2

Remark

- It is not even known whether γ is irrational.
- γ is called Euler-Mascheroni constant, and $\gamma \simeq 0,5772\ldots$.

Proof. We know

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

thus

$$n\log\left(1+\frac{1}{n}\right) < 1 < (n+1)\log\left(1+\frac{1}{n}\right),$$

and consequently

$$\log\left(\frac{n+1}{n}\right) < \frac{1}{n},$$

$$\log\left(\frac{n+1}{n}\right) > \frac{1}{n+1}.$$

Proof: 2/2

Thus

$$a_{n+1} - a_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) - \sum_{k=1}^{n-1} \frac{1}{k} + \log(n) = \frac{1}{n} - \log\left(\frac{n+1}{n}\right) > 0.$$

Hence $(a_n)_{n\in\mathbb{N}}$ is increasing. Similarly,

$$b_{n+1} - b_n = \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right) < 0,$$

thus $(b_n)_{n\in\mathbb{N}}$ is decreasing. Also it is clear

$$a_1 \leq a_n \leq b_n \leq b_1.$$

Thus by the (MCT) the limits exist

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \gamma,$$

since $b_n = a_n + \frac{1}{n}$. □

Bolzano–Weierstrass theorem

Bolzano–Weierstrass theorem

Every bounded sequence contains convergent subsequence.

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be bounded. Then there is $M > 0$ such that

$$|a_n| \leq M \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus $a_n \in [-M, M]$ for all $n \in \mathbb{N}$.

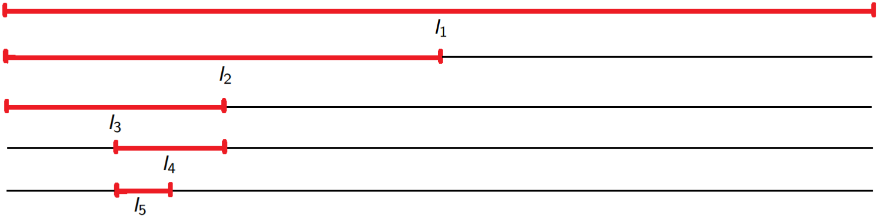
- **Step 1.** Divide $[-M, M]$ into two closed intervals $[-M, 0]$, $[0, M]$. We can assume (wlog) that $I_1 = [0, M]$ contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$. Moreover, the length of I_1 is M.
- **Step 2.** Divide I_1 into two closed intervals of the same length and select the one which contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$. Call it $I_2 \subset I_1$ and note that has length $\frac{M}{2}$.

Proof: 1/2

- **Step 3.** Proceeding inductively as above we obtain a sequence of decreasing closed intervals

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset \dots$$

where each I_k contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$ and has length $\frac{M}{2^{k-1}}$.



Proof: 2/2

- **Step 4.** By the Nested Intervals Property $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$. In fact,

$$\bigcap_{k=1}^{\infty} I_k = \{x\} \quad \text{for some} \quad x \in \mathbb{R}.$$

  Now for each $k \in \mathbb{N}$ select an element $a_{n_k} \in I_k$ so that

$$n_1 < n_2 < \ldots < n_k < \ldots$$

  where a_{n_1} is any element of I_1.

- **Step 5.** Let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ so that

$$\frac{M}{2^{k-1}} < \varepsilon \quad \text{for} \quad k \geq N_\varepsilon.$$

  Then for every $k \geq N_\varepsilon$ we have

$$|a_{n_k} - x| \leq \frac{M}{2^{k-1}} < \varepsilon,$$

  thus $\lim_{n\to\infty} a_{n_k} = x$. □

Bolzano–Weierstrass theorem - example

Example

Let us consider a sequence

$$a_n = (-1)^n.$$

It in **NOT** convergent, but the subsequence $(-1)^{2n} = 1$ converges to 1.