Lesson 14

Completenes
Infinite Series and Euler's numbers

MATH 311, Section 4, FALL 2022

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Cauchy sequences

Cauchy sequences

A sequence $(a_n)_{n\in\mathbb{N}}$ is called **a Cauchy sequence** if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that whenever $m, n \geq N_\varepsilon$ it follows

$$|a_n - a_m| < \varepsilon.$$

Convergent sequences

Recall that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to $a \in \mathbb{R}$ if for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that whenever $n \geq N_\varepsilon$ if follows

$$|a_n - a| < \varepsilon.$$

Convergent sequences are Cauchy

Theorem

Every convergent sequence is a Cauchy sequence.

Proof. Let $\varepsilon > 0$ be given. If

$$\lim_{n\to\infty} x_n = x,$$

then there is $N_\varepsilon \in \mathbb{N}$ so that $n \geq N_\varepsilon$ implies

$$|x_n - x| < \frac{\varepsilon}{2}.$$

Thus for $n, m \geq N_\varepsilon$ we obtain

$$|x_m - x_n| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is completed. □

Bolzano–Weierstrass theorem

Lemma

Cauchy sequences are bounded.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be Cauchy. Given $\varepsilon = 1$ there is $N \in \mathbb{N}$ so that if $n, m \geq N$ then $|x_n - x_m| < 1$. Thus

$$|x_n| \leq |x_N| + 1.$$

Taking

$$M = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$$

we conclude $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

Bolzano–Weierstrass theorem

Every bounded sequence contains convergent subsequence.

Cauchy Criterion

Cauchy Criterion

A sequence $(x_n)_{n\in\mathbb{N}}$ converges iff it is a Cauchy sequence.

Proof: The implication ($\Longrightarrow$) has already been proved. For the reverse implication ($\Longleftarrow$) assume that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. By the previous lemma the sequence is bounded. Hence by **the Bolzano–Weierstrass** theorem there is $(n_k)_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} x_{n_k} = x \quad \text{for some} \quad x \in \mathbb{R} \qquad (*).$$

Let $\varepsilon > 0$ be given. Then there is $N_\varepsilon \in \mathbb{N}$ so that $n, m \geq N_\varepsilon$ implies $|x_n - x_m| < \frac{\varepsilon}{2}$. By (*) we can choose $n_k \in \mathbb{N}$ so that $n_k \geq N_\varepsilon$ and

$$|x_{n_k} - x| < \frac{\varepsilon}{2}.$$

Then for $n \geq N_\varepsilon$ and the triangle inequality

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Series

Definition

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges** to $A \in \mathbb{R}$ and write $\sum_{n=1}^{\infty} a_n = A$ if the associated sequence of its **partial sums**

$$s_n = \sum_{k=1}^{n} a_k \xrightarrow[n\to\infty]{} A.$$

If $(s_n)_{n\in\mathbb{N}}$ diverges the series $\sum_{n=1}^{\infty} a_n$ is said **to diverge**.

Remark

- Saying that the series $\sum_{n=1}^{\infty} a_n$ converges we understand that $|\sum_{k=1}^{\infty} a_k| < \infty$.
- Saying that the series $\sum_{n=1}^{\infty} a_n$ diverges we understand that $|\sum_{k=1}^{\infty} a_k| = \infty$.

Algebraic limit theorem for series

Algebraic limit theorem for series

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ then

$$\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) = \alpha A + \beta B.$$

Proof. Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$. We know that

$$\lim_{n\to\infty} A_n = A, \quad \text{and} \quad \lim_{n\to\infty} B_n = B,$$

so

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n}(\alpha a_k + \beta b_k) &= \lim_{n\to\infty} \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k \\ &= \alpha \lim_{n\to\infty} A_n + \beta \lim_{n\to\infty} B_n = \alpha A + \beta B.\end{aligned}$$

□

Geometric series

Geometric series

If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, the series diverges.

Solution. If $x < 1$, then

$$s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

and the result follows if we let $n \to \infty$.

For $x \geq 1$ note that

$$\underbrace{1 + 1 + \ldots + 1}_{n} \leq s_n.$$

We have $\lim_{n\to\infty} n = +\infty$, thus $\lim_{n\to\infty} s_n = +\infty$.

Cauchy Criterion for Series

Theorem

The series $\sum_{k=1}^{\infty} a_k$ converges iff for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that whenever $n > m \geq N_\varepsilon$ it follows

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and we show that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Observe that whenever $n > m \geq N_\varepsilon$ then

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

We now apply the **Cauchy Criterion for sequences** and we are done. □

Theorem

Theorem

If the series $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $\varepsilon > 0$ be given. Apply the previous theorem with $m = n - 1$, then

$$|a_n| = |s_n - s_{n-1}| < \varepsilon$$

whenever $n > N_\varepsilon$, and we are done. □

Remark

But $\lim_{n\to\infty} a_n = 0$ does not imply $|\sum_{k=1}^{\infty} a_k| < \infty$.

- Consider $a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$, but $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Example

Exercise

Determine if the series

$$\sum_{n=1}^{\infty}(-1)^n \left(1 - \frac{1}{n^3}\right)^{n^2}$$

diverges or converges.

Solution. Since $(n^3)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty} \left(1 - \frac{1}{n^3}\right)^{n^3} = e^{-1},$$

hence $\lim_{n\to\infty} \left(1 - \frac{1}{n^3}\right)^{n^2} = 1$, and the limit $\lim_{n\to\infty}(-1)^n \left(1 - \frac{1}{n^3}\right)^{n^2}$ does not exist, so the series **diverges**. □

Comparison test

Comparison test

Assume that sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ satisfy

$$0 \le a_k \le b_k \quad \text{for all} \quad k \in \mathbb{N}.$$

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Both statements follows from the Cauchy Criterion for series:

$$\left|\sum_{k=m+1}^{n} a_k\right| \le \left|\sum_{k=m+1}^{n} b_k\right|.$$

This completes the proof. □

Example

Exercise

Determine if the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n} + 15}$$

diverges or converges.

Solution. For all $n \in \mathbb{N}$ we have

$$\frac{1}{n^2 + \sqrt{n} + 15} \leq \frac{1}{n^2}, \qquad \text{thus}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n} + 15} < \infty. \qquad \square$$

Example

Exercise

Determine if the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}+\sqrt{n}+1}$$

diverges or converges.

Solution. For all $n \in \mathbb{N}$ we have

$$\frac{1}{\sqrt[3]{n}+\sqrt{n}+1} \geq \frac{1}{3\sqrt{n}}, \qquad \text{thus}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}+\sqrt{n}+1} = \infty. \qquad \square$$

Theorem

Theorem

A series of nonnegative terms $a_k \geq 0$ converges iff its partial sums form a bounded sequence.

Proof. If $\sum_{k=1}^{\infty} a_k < \infty$ one sees that

$$s_N = \sum_{k=1}^{N} a_k \leq M = \sum_{k=1}^{\infty} a_k < \infty.$$

Conversely, we also know that $s_N \leq s_{N+1} \leq M$ for all $N \in \mathbb{N}$. Then the limit

$$\lim_{N\to\infty} s_N$$

exists by the (MCT). □

Cauchy Condensation Test

We have seen that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty, \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Cauchy Condensation Test

Suppose that $(b_n)_{n\in\mathbb{N}}$ is decreasing and $b_n \geq 0$ for all $n \in \mathbb{N}$. Then the series

$$\sum_{n=1}^{\infty} b_n < \infty \quad \text{converges}$$

iff the series

$$\sum_{n=1}^{\infty} 2^n b_{2^n} < \infty \quad \text{converges.}$$

Corollary

Corollary

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \text{iff} \quad p > 1$$

Proof. The sequence $b_n = \frac{1}{n^p}$ is decreasing and $b_n \geq 0$ for all $n \in \mathbb{N}$. By the Cauchy condensation test we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \iff \quad \sum_{n=0}^{\infty} \frac{2^n}{2^{pn}} < \infty.$$

But the latter converges provided that

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{pn}} = \sum_{n=0}^{\infty} 2^{(1-p)n} = \frac{1}{1 - \frac{1}{2^{p-1}}} < \infty \quad \iff \quad p > 1. \quad \square$$

Euler's number e

Theorem

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Proof. Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. Then

1. $s_n < s_{n+1}$ for all $n \in \mathbb{N}$,
2. $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} < 2 + \sum_{k=2} \frac{1}{2^{k-1}} < 3.$

Thus the limit $\lim_{n\to\infty} s_n$ exists.

Let $t_n = \left(1 + \frac{1}{n}\right)^n$, then $\lim_{n\to\infty} t_n = e$. By the binomial theorem

$$t_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}.$$

Proof: 1/2

Then

$$\begin{aligned} t_n &= \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{1}{n^k} \\ &= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots \\ &+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot \ldots \cdot\left(1-\frac{n-1}{n}\right) \le \sum_{k=0}^{n} \frac{1}{k!} = s_n. \end{aligned}$$

Thus

$$e = \lim_{n\to\infty} t_n \le \lim_{n\to\infty} s_n.$$

Proof: 2/2

Next if $n \geq m$

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

Let $n \to \infty$ keeping m fixed, we get

$$e = \lim_{n\to\infty} t_n \geq \sum_{k=0}^{m} \frac{1}{k!}.$$

Letting $m \to \infty$ we see $\lim_{m\to\infty} s_m \leq e$.

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} = e.$$

This completes the proof of the theorem. □

Remark

We have $s_n = \sum_{k=0}^{n} \frac{1}{k!} < e$ for all $n \in \mathbb{N}$. Indeed

$$
\begin{aligned}
e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \\
&= \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots \right) \\
&< \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots \right) \\
&\le \frac{1}{(n+1)!} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{(n+1)!} \frac{n+1}{n} = \frac{1}{n!n}.
\end{aligned}
$$

Hence we conclude

The error estimate (*)

$$0 < e - s_n < \frac{1}{n!n}.$$

Euler's number e is irrational

Theorem

The Euler number e is irrational.

Proof. Suppose e is rational. Then $e = \frac{p}{q}$ where $p, q \in \mathbb{N}$. By (*) we have

$$0 < q!(e - s_q) < \frac{1}{q}.$$

By our assumption

$$q!e \in \mathbb{N} \quad \text{is an integer.}$$

Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\right) \in \mathbb{N},$$

we see $q!(e - s_q) \in \mathbb{N}$, but if $q > 1$ and this is impossible since

$$0 < q!(e - s_q) < 1/q < 1.$$

Hence e must be irrational. □

Euler's number and the exponential function

We know that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

for any $x \in \mathbb{R}$.

Also

$$\sum_{n=0}^{\infty}\frac{1}{n!} = e.$$

Theorem

Let $x \in \mathbb{R}$, then

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x.$$

Proof: 1/2

Proof. Let $S_n = \sum_{k=0}^{n} \frac{x^k}{k!}$, then by the binomial theorem we may write

$$\left| S_n - \left(1 + \frac{x}{n}\right)^n \right| = \left| \sum_{k=2}^{n} \left(1 - \left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right)\right) \frac{x^k}{k!} \right|$$
$$\leq \sum_{k=2}^{n} \left(1 - \left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right)\right) \frac{|x|^k}{k!}.$$

Let us also note that

$$\left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{n}\right) \geq 1 - \sum_{j=1}^{k-1} \frac{j}{n} = 1 - \frac{k(k-1)}{2n}$$

for $2 \leq k \leq n$.

Proof: 2/2

Thus

$$\left|S_n - \left(1+\frac{x}{n}\right)^n\right| \le \sum_{k=2}^{n} \frac{k(k-1)}{2n}\frac{|x|^k}{k!} = \frac{1}{2n}\sum_{k=2}^{n}\frac{|x|^k}{(k-2)!}.$$

Using the Stolz theorem

$$\lim_{n\to\infty}\frac{1}{2n}\sum_{k=2}^{n}\frac{|x|^k}{(k-2)!} = \lim_{n\to\infty}\frac{\sum_{k=2}^{n+1}\frac{|x|^k}{(k-2)!} - \sum_{k=2}^{n}\frac{|x|^k}{(k-2)!}}{(2n+2)-2n}$$

$$= \lim_{n\to\infty}\frac{1}{2}\frac{|x|^{n+1}}{(n-1)!} = 0.$$

Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

as desired. □