Lesson 15

Exponential function and logarithm
Upper and Lower limits
Properties of infinite series and Abel Summation formula

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Exponential function

We know that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x \quad \text{for any} \quad x\in\mathbb{R}.$$

Exponential function

The function $E:\mathbb{R}\to(0,\infty)$ defined by $E(x)=e^x$ is called **the exponential function**.

Properties of exponential function

1. For all $x, y\in\mathbb{R}$ one has
$$e^{x+y}=e^x e^y.$$
2. If $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} e^{a_n}=e^a$.
3. E is one-to-one and onto. Thus the inverse for E exists.

Natural logarithm

Natural logarithm

The inverse of E exists. It will be denoted by $E^{-1} : (0, \infty) \to \mathbb{R}$,

$$E^{-1}(x) = \ln(x) = \log(x)$$

and it is called **the natural logarithm**.

Simple properties of natural logarithm

1. $\log(x)$ is increasing.
2. For $x, y \in (0, \infty)$ we have
$$\log(xy) = \log(x) + \log(y).$$
3. We also have $x^\alpha = e^{\alpha \log(x)}$ for all $\alpha \in \mathbb{R}$.

Proposition

Proposition

For $x > 0$ we have

$$\frac{x}{x+2} < \log(x+1) < x.$$

Proof. We prove that for $0 < x < m$ with $m \in \mathbb{N}$, we have

$$\left(1+\frac{x}{n}\right)^{n} < e^{x} < \left(1+\frac{x}{n}\right)^{n+m}.$$

thus

$$n\log\left(1+\frac{x}{n}\right) < x < (n+m)\log\left(1+\frac{x}{n}\right).$$

Hence

$$\frac{x}{n+m} < \log\left(1+\frac{x}{n}\right) < \frac{x}{n} \quad \text{if} \quad m > x.$$

Proof:

Taking $n = 1$ we obtain

$$\log(1+x) < x \quad \text{for all} \quad x > 0.$$

Now set $m = \lfloor x \rfloor + 1 > x$, then

$$\log\left(1 + \frac{x}{n}\right) > \frac{\frac{x}{n}}{2 + \frac{x}{n}}.$$

Thus for $n = 1$ we obtain

$$\log(1+x) > \frac{x}{2+x}.$$

□

Remark

In fact, for every $x > 0$ the following inequality holds

$$\frac{x}{x+1} < \log(x+1) < x.$$

Euler–Mascheroni constant

Divergence of harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Theorem

The sequences

$$a_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log(n) \quad \text{and} \quad b_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n)$$

are increasing and decreasing respectively and bounded, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \gamma.$$

where γ is known as **the Euler (or Euler–Mascheroni) constant**.

Proof: 1/2

Remark

- It is not even known whether γ is irrational.
- γ is called Euler-Mascheroni constant, and $\gamma \simeq 0,5772\ldots$.

Proof. We know

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

thus

$$n\log\left(1+\frac{1}{n}\right) < 1 < (n+1)\log\left(1+\frac{1}{n}\right),$$

and consequently

$$\log\left(\frac{n+1}{n}\right) < \frac{1}{n},$$

$$\log\left(\frac{n+1}{n}\right) > \frac{1}{n+1}.$$

Proof: 2/2

Thus

$$a_{n+1} - a_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) - \sum_{k=1}^{n-1} \frac{1}{k} + \log(n) = \frac{1}{n} - \log\left(\frac{n+1}{n}\right) > 0.$$

Hence $(a_n)_{n\in\mathbb{N}}$ is increasing. Similarly,

$$b_{n+1} - b_n = \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right) < 0,$$

thus $(b_n)_{n\in\mathbb{N}}$ is decreasing. Also it is clear

$$a_1 \le a_n \le b_n \le b_1.$$

Thus by the (MCT) the limits exist

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \gamma,$$

since $b_n = a_n + \frac{1}{n}$. □

Upper and lower limits

Upper limit and lower limit

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

- **The upper limit** is defined by

$$\limsup_{n\to\infty} s_n = \inf_{k\geq 1} \sup_{n\geq k} s_n.$$

- **The lower limit** is defined by

$$\liminf_{n\to\infty} s_n = \sup_{k\geq 1} \inf_{n\geq k} s_n.$$

Proposition

Proposition

For a sequence $(s_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, the upper and lower limits always exist.

Proof. Let $\alpha_k = \sup_{n\geq k} s_n$. Then $\alpha_{k+1} \leq \alpha_k$ and

$$\limsup_{n\to\infty} s_n = \inf_{k\geq 1} \sup_{n\geq k} s_n \underbrace{=}_{(MCT)} \lim_{n\to\infty} \alpha_k \text{ (possible infinite!)}.$$

If $\beta_k = \inf_{n\geq k} s_n$, then $\beta_k \leq \beta_{k+1}$ and

$$\liminf_{n\to\infty} s_n = \sup_{k\geq 1} \inf_{n\geq k} s_n \underbrace{=}_{(MCT)} \lim_{n\to\infty} \beta_k \text{ (possible infinite!)}.$$

□

Remark

Useful remarks

We always have

$$\beta_k = \inf_{n \geq k} s_n \leq \sup_{n \geq k} s_n = \alpha_k.$$

thus

$$\liminf_{n \to \infty} a_n = \lim_{k \to \infty} \beta_k \leq \lim_{k \to \infty} \alpha_k = \limsup_{n \to \infty} s_n.$$

Proposition

If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$ then $\lim_{n\to\infty} s_n = L$.

Proof. If $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$, then

$$\alpha_k = \inf_{n \geq k} s_n \leq s_k \leq \sup_{n \geq k} s_n = \beta_k$$

and $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = L$, thus $\lim_{n\to\infty} s_n = L$. □

Examples 1/3

Example 1

Consider $a_n = (-1)^n \frac{n+1}{n}$. Let

$$\beta_n = \sup\left\{(-1)^n \frac{n+1}{n}, (-1)^{n+1}\frac{n+2}{n+1}, \dots\right\},$$

then

$$\beta_n = \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is even,} \\ \frac{n+2}{n+1} & \text{if } n \text{ is odd.} \end{cases}$$

Thus $\lim_{n\to\infty} \beta_n = 1$. Therefore

$$\limsup_{n\to\infty} a_n = 1.$$

Similarly

$$\liminf_{n\to\infty} a_n = -1.$$

Examples 2/3

Example 2

Let

$$a_n = \begin{cases} 0 \text{ if } n \text{ is odd,} \\ 1 \text{ if } n \text{ is even.} \end{cases}$$

Then

$$\beta_n = \sup\{a_m \;:\; m \geq n\} = 1,$$
$$\alpha_n = \inf\{a_m \;:\; m \geq n\} = 0.$$

Therefore

$$\limsup_{n\to\infty} a_n = 1,$$
$$\liminf_{n\to\infty} a_n = 0.$$

Examples 3/3

Example 3

Let $a_n = \frac{1}{n}$. Then

$$\beta_n = \sup\left\{\frac{1}{m} \ : \ m \geq n\right\} = \frac{1}{n},$$

so $\lim_{n\to\infty} \beta_n = 0$. Similarly

$$\alpha_n = \inf\left\{\frac{1}{m} \ : \ m \geq n\right\} = 0,$$

so $\lim_{n\to\infty} \alpha_n = 0$. Thus

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 0.$$

Absolute convergence

Absolute convergence

The series $\sum_{n=1}^{\infty} a_n$ is said **to converge absolutely** if the series

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

converges.

Theorem

If $\sum_{n=1}^{\infty} |a_n| < \infty$, then $|\sum_{n=1}^{\infty} a_n| < \infty$.

Proof. The claim follows from the Cauchy Criterion, since

$$\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k|$$

and we are done. □

Conditional convergence

Conditional convergence

If the series $\sum_{n=1}^{\infty} a_n$ converges but

$$\sum_{n=1}^{\infty} |a_n| = \infty$$

diverges then we say that $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Example 1

For series with positive terms, absolute convergence is the same as convergence.

Example 2

$\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ converges absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

Anharmonic series

Anharmonic series

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges conditionally.

It is easy to see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

To prove $\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right| < \infty$ we will show a more general result.

Summation by parts (Abel summation formula)

Abel summation formula

Given two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ set

$$A_n = \sum_{k=0}^{n} a_k \quad \text{for} \quad n \geq 0, \quad \text{and} \quad A_{-1} = 0.$$

Then if $0 \leq p \leq q$ one has

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n \left(b_n - b_{n+1}\right) + A_q b_q - A_{p-1} b_p.$$

Proof of Abel summation formula

Proof: Note that

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} \underbrace{(A_n - A_{n-1})}_{a_n} b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

The proof follows. □

Theorem

Dirichlet's test

Suppose that

(a) The partial sums $A_n = \sum_{k=1}^{n} a_k$ of $(a_n)_{n\in\mathbb{N}}$ form a bounded sequence.

(b) $b_0 \geq b_1 \geq b_2 \geq b_3 \geq \ldots$,

(c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof. Choose $M \geq 0$ so that $|A_n| \leq M$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ so that

$$b_{N_\varepsilon} < \frac{\varepsilon}{2M},$$

since $\lim_{n\to\infty} b_n = 0$.

Proof

For $N_\varepsilon \le p \le q$, by the summation by parts formula, one has

$$\left|\sum_{n=p}^{q} a_n b_n\right| = \left|\sum_{n=p}^{q-1} A_n\left(b_n - b_{n+1}\right) + A_q b_q - A_{p-1} b_p\right|$$
$$\le \left|\sum_{n=p}^{q-1} A_n\left(b_n - b_{n+1}\right)\right| + |A_q b_q| + |A_{p-1} b_p|$$
$$\le M\sum_{n=p}^{q-1} |(b_n - b_{n+1})| + M b_q + M b_p \le 2Mb_p \le 2Mb_{N_\varepsilon} < \varepsilon.$$

since

$$b_p - b_q = \sum_{n=p}^{q-1} |(b_n - b_{n+1})| = \sum_{n=p}^{q-1} (b_n - b_{n+1})$$
$$= (b_p - b_{p+1}) + (b_{p+1} - b_{p+2}) + (b_{p+2} - b_{p+3}) + \ldots + b_{q-1} - b_q. \quad \square$$

Anharmonic series

We now show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Let

$$a_n = (-1)^n, \quad \text{and} \quad b_n = \frac{1}{n}$$

in the previous theorem. We see that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right| = \left| \sum_{n=1}^{\infty} a_n b_n \right| < \infty$$

since

$$|A_n| = \left| \sum_{k=1}^{n} (-1)^k \right| \leq 1.$$

Alternating Series Test

A more general result can be proved:

Alternating Series Test

Let $(a_n)_{n\in\mathbb{N}}$ be such that

1. (i) $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots$,
2. (ii) $\lim_{n\to\infty} a_n = 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Proof. We apply the previous theorem.

Example

Exercise

Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges and converges absolutely.

Solution. Let $a_n = \frac{1}{\sqrt{n^2+1}}$.

- We have
$$a_n \geq \frac{1}{\sqrt{4n^2}} = \frac{1}{2n},$$
so the series **does not converges absolutely**, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- On the other hand, we have
$$a_n \geq a_{n+1} \quad \text{and} \quad \lim_{n\to\infty} a_n = 0,$$
so the assumptions of the previous theorem are satisfied. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ **converges conditionally**. □

Root test

Root test

Given $\sum_{n=1}^{\infty} a_n$ set

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

- (a) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- (b) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- (c) If $\alpha = 1$, no information.

Proof. If $\alpha < 1$ we can choose β so that $\alpha < \beta < 1$ and the integer $N \in \mathbb{N}$ so that

$$\sqrt[n]{|a_n|} < \beta \quad \text{for all} \quad n \geq N,$$

since

$$\alpha = \inf_{k\geq 1} \sup_{n\geq k} \sqrt[n]{|a_n|} < \beta.$$

Proof

- For $n \geq N$ we have $|a_n| < \beta^n$, but $\beta < 1$, thus $\sum_{n=1}^{\infty} \beta^n$ converges and the comparision test implies that $\sum_{n=1}^{\infty} a_n$ converges as well.
- If $\alpha > 1$ then there is $(n_k)_{k\in\mathbb{N}}$ so that

$$\left|a_{n_k}\right|^{1/n_k} \xrightarrow[k\to\infty]{} \alpha.$$

Hence $|a_n| > 1$ holds for infinitely many values of $n \in \mathbb{N}$, so that the condition $a_n \xrightarrow[n\to\infty]{} 0$ necessary for convergence $\sum_{n=1}^{\infty} a_n$ does not hold.
- To prove (c) note that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{and} \quad \sqrt[n]{n} \xrightarrow[n\to\infty]{} 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad \text{and} \quad \sqrt[n]{n^2} \xrightarrow[n\to\infty]{} 1.$$

This completes the proof. □

Examples

Example 1

$$\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n < \infty,$$

since

$$\sqrt[n]{\frac{e^n}{n^n}} = \frac{e}{n} \xrightarrow[n\to\infty]{} 0.$$

Example 2

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} < \infty,$$

since

$$\sqrt[n]{\frac{n^2}{2^n}} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

Ratio test

Ratio test

The series $\sum_{n=1}^{\infty} a_n$

(a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,

(b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for all $n \geq n_0$ for some fixed $n_0 \in \mathbb{N}$.

Proof. If (a) holds we can find $\beta < 1$ and $n \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for all} \quad n \geq N.$$

Proof

- In particular, for $p \in \mathbb{N}$, one has

$$
\begin{aligned}
|a_{n+p}| &= |a_{n+p-1}| \frac{|a_{n+p}|}{|a_{n+p-1}|} \\
&< \beta |a_{n+p-1}| \\
&< \beta^2 |a_{n+p-2}| < \ldots < \\
&< \beta^p |a_n|.
\end{aligned}
$$

- Thus $|a_{N+p}| < \beta^p |a_N|$ and

$$|a_n| < |a_N| \beta^{-N} \beta^n \quad \text{for all} \quad n \geq N.$$

- The claim follows from the comparison test since $\sum_{n=1}^{\infty} \beta^n < \infty$ whenever $\beta < 1$.
- If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$ then $a_n \xrightarrow[n \to \infty]{} 0$ does not hold. □

Remark and example

Remark

As before $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ is useless:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{and} \quad \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad \text{and} \quad \frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^2 \xrightarrow[n\to\infty]{} 1.$$

Example

$\sum_{n=1}^{\infty} \frac{n!}{n^n} < \infty$, since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e} < 1.$$

Rearrangements

Rearrangement

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence in which every positive integer appears once and only once. Setting

$$a'_n = a_{k_n}$$

we say that $\sum_{n=1}^{\infty} a'_n$ is **rearrangement** of $\sum_{n=1}^{\infty} a_n$.

Example

- Consider the convergent series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} \underbrace{- \frac{1}{4} + \frac{1}{5}}_{<0} \underbrace{- \frac{1}{6} + \frac{1}{7}}_{<0} - \dots$$

Example

- Consider also a rearrangement S' of S given by:

$$S' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots + $$
$$= \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right)$$

- Observe that $S < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

- If S'_n is the partial sum of S' then

$$S'_3 < S'_6 < S'_9 < \ldots$$

hence $\limsup_{n\to\infty} S'_n > S'_3 = \frac{5}{6}$.
- Thus S' **does not converge** to $S < \frac{5}{6}$.

Theorem

Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series that converges unconditionally. Suppose that

$$-\infty \leq \alpha \leq \beta \leq +\infty.$$

Then there exists a reaarangement $\sum_{n=0}^{\infty} a'_n$ with partial sums s'_n so that

$$\liminf_{n\to\infty} s'_n = \alpha, \quad \text{and} \quad \limsup_{n\to\infty} s'_n = \beta.$$

Theorem

If $\sum_{n=1}^{\infty} |a_n| < \infty$, then every rearrangement of $\sum_{n=1}^{\infty} a_n$ converge to the same limit.

Proof. Let $\sum_{n=1}^{\infty} a'_n$ be a rearrangement of $\sum_{n=1}^{\infty} a_n$ with partial sums s'_n. Given $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that $m \geq n \geq N_\varepsilon$ implies

$$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

Now choose $p \in \mathbb{N}$ such that

$$\{1, 2, \ldots, N_\varepsilon\} \subseteq \{k_1, k_2, \ldots, k_p\}.$$

If $n > p$ then the numbers $a_1, \ldots, a_N$ will cancel in the difference $s_n - s'_n$ so that

$$|s_n - s'_n| < \varepsilon.$$

Hence s'_n converges to the same limit as $(s_n)_{n \in \mathbb{N}}$. □