# Lesson 17

## Complete spaces and Compact sets

MATH 311, Section 4, FALL 2022

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# Proposition

**Proposition**

Let $(X, \rho)$ be a metric space, $E \subseteq X$ is closed iff for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ such that $x_n \xrightarrow[n\to\infty]{} x \in X$ we have $x \in E$.

**Proof.** **($\Longrightarrow$)** Suppose that $E$ is closed and consider $(x_n)_{n\in\mathbb{N}} \subseteq E$ such that $x_n \xrightarrow[n\to\infty]{} x \in X$. We have to show that $x \in E$. Observe that

$$B(x, r) \cap E \neq \emptyset \quad \text{for any} \quad r > 0.$$

But $x_n \xrightarrow[n\to\infty]{} x$ iff $x_n \in B(x, r)$ for all but finitely many $n \in \mathbb{N}$, and consequently we conclude that $x \in \mathrm{cl}\,(E)$, hence $x \in E$.

**($\Longleftarrow$)** Conversely, if $x \in \mathrm{cl}\,(E)$ then there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ so that $x_n \xrightarrow[n\to\infty]{} x \in \mathrm{cl}\,(E) \subseteq X$ thus by our assumption $x \in E$. □

# Theorem

**Theorem**

The subsequential limits of a sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $(X,\rho)$ form a closed subset of $X$.

**Proof.** Let $E^*$ be the set of subsequential limits of $(x_n)_{n\in\mathbb{N}}$ and let $q$ be an accumulation point of $E^*$. We will show that $q \in E^*$.

- Choose $n_1 \in \mathbb{N}$ so that $E^* \ni x_{n_1} \neq q$ (if no such point exists then $E^*$ has only one point and there is nothing to prove). Set

$$\delta = \rho(x_{n_1}, q) > 0.$$

- Suppose that $n_1, \ldots, n_{i-1}$ have been chosen. Since $q$ is an accumulation point of $E^*$ there is $x \in E^*$ so that

$$\rho(x,q) < \delta 2^{-i-1}.$$

Since $x \in E^*$ there is $n_i > n_{i-1}$ such that $\rho(x, x_{n_i}) < \delta 2^{-i-1}$.

# Proof

- Hence, by the triangle inequality

$$\rho(q, x_{n_i}) \leq \rho(q, x) + \rho(x, x_{n_i}) < \delta 2^{-i-1} + \delta 2^{-i-1} = \delta 2^{-i}.$$

- This means that $(x_{n_i})_{i \in \mathbb{N}}$ converges to $q$, i.e.

$$\lim_{i \to \infty} x_{n_i} = q \quad \iff \quad \lim_{i \to \infty} \rho(x_{n_i}, 0) = 0$$

  thus $q \in E^*$.

- In fact, we have shown that

$$\mathrm{acc}\,(E^*) \subseteq E^*,$$

  which means that $E^*$ is closed. □

# Cauchy sequences

## Cauchy sequence

A sequence $(x_n)_{n\in\mathbb{N}}$ is a metric space $(X,\rho)$ is said to be **a Cauchy sequence** if for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

$$m, n \geq N_\varepsilon \quad \text{implies} \quad \rho(x_m, x_n) < \varepsilon.$$

## Complete spaces

A subset of a metric space $(X,\rho)$ is called **complete** if every Cauchy sequence in $E$ converges and its limit is in $E$.

# Complete spaces - examples

### Example 1

The set of real numbers $\mathbb{R}$ is complete.

### Example 2

The open unit interval $(0, 1)$ is not complete space in $\mathbb{R}$.

- Indeed, let $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$, then $x_n \in (0, 1)$ and $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(0, 1)$, but 0, which is the limit of $(x_n)_{n\in\mathbb{N}}$ is not in $(0, 1)$.

### Example 3

$[0, 1]$ is complete space in $\mathbb{R}$.

# Some facts

**Fact 1**

If $(x_n)_{n\in\mathbb{N}}$ is Cauchy in a metric space $(X,\rho)$ then $(x_n)_{n\in\mathbb{N}}$ is bounded.

**Fact 2**

If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a metric space $(X,\rho)$ and $\lim_{k\to\infty}\rho(x_{n_k},x)=0$ for some $(x_{n_k})_{k\in\mathbb{N}}$, then

$$\lim_{n\to\infty}\rho(x_n,x)=0.$$

**Proposition**

A closed subset of a complete metric space is complete and a complete subset of an arbitrary metric space is closed.

# Proof of the Proposition

**Proof.** If $(X, \rho)$ is complete, $E \subseteq X$ is closed and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $E$, then $(x_n)_{n\in\mathbb{N}}$ has a limit in $X$. But $\mathrm{cl}\,(E) = E$, thus $x \in \mathrm{cl}\,(E)$, so $x \in E$.

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If $E \subseteq X$ is complete and $x \in \mathrm{cl}\,(E)$ then we know that there exists $(x_n)_{n\in\mathbb{N}} \subseteq E$ converging to $x$. But $(x_n)_{n\in\mathbb{N}}$ is Cauchy so its limit lies in $E$, thus $\mathrm{cl}\,(E) = E$ as desired. □

**Remark**

In the second part of the proof we have used the fact that if $(x_n)_{n\in\mathbb{N}}$ converges (say to $x$ in a metric space $(X, \rho)$) then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

# Cantor intersection theorem

## Cantor intersection theorem

A metric space is complete iff for every decreasing sequence

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

of nonempty closed sets in $X$ with $\mathrm{diam}\,(F_n) \xrightarrow[n\to\infty]{} 0$, one has

$$\bigcap_{n\in\mathbb{N}} F_n = \{x_0\} \quad \text{for some} \quad x_0 \in X.$$

# Proof ($\Longrightarrow$):

Assume that $(X, \rho)$ is complete.

- Let $(F_n)_{n\in\mathbb{N}}$ be such that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ and $\operatorname{diam}(F_n) \xrightarrow[n\to\infty]{} 0$.
- Choose $x_n \in F_n$, let $\varepsilon > 0$ and pick $N_\varepsilon \in \mathbb{N}$ such that $\operatorname{diam}(F_n) < \varepsilon$ for all $n \geq N_\varepsilon$. Note that for $n \geq m \geq N_\varepsilon$ we have

$$x_n \in F_n \subseteq F_m,$$

  so

$$\rho(x_n, x_m) \leq \operatorname{diam}(F_m) < \varepsilon.$$

- This ensures that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and, consequently, converges to some $x_0 \in X$. Since each $F_n$ is closed then $x_0 \in F_n$ for all $n \in \mathbb{N}$, thus $x_0 \in \bigcap_{n\in\mathbb{N}} F_n$.
- Suppose there is $y \neq x_0$ so that $y \in \bigcap_{n\in\mathbb{N}} F_n$, then

$$0 < \rho(x_0, y) \leq \operatorname{diam}(F_n) \xrightarrow[n\to\infty]{} 0,$$

  contradiction. Thus $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$.

# Proof ($\Longleftarrow$):

To prove the converse implication assume that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

- Let

$$F_n = \mathrm{cl}\,\left(\{x_m \;:\; m \geq n\}\right).$$

**Fact**

$$\mathrm{diam}\,(E) = \mathrm{diam}\,(\mathrm{cl}\,(E)).$$

- We see that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$ and

$$\mathrm{diam}\,(F_n) = \mathrm{diam}\,\left(\{x_m \;:\; m \geq n\}\right) \xrightarrow[n\to\infty]{} 0.$$

  Thus $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$ for some $x_0 \in X$.
- Finally, we conclude $\lim_{n\to\infty} \rho(x_n, x_0) = 0$ as desired. □

# Coverings and compact sets

## Coverings

Let $(X, \rho)$ be a metric space.

- If $E \subseteq X$ and $(V_\alpha)_{\alpha \in A}$ is a family of sets such that $E \subseteq \bigcup_{\alpha \in A} V_\alpha$, then $(V_\alpha)_{\alpha \in A}$ is called **a cover** of $E$ and $E$ is said to be **covered** by the $V_\alpha$'s.
- If additionally each $V_\alpha$ is open $(V_\alpha)_{\alpha \in A}$ is called **an open cover of $E$.**

## Heine–Borel property

A subset $K$ of a metric space $(X, \rho)$ is said to be **compact** if every open cover of $K$ contains a finite subcover. More explicitly, if $(V_\alpha)_{\alpha \in A}$ is an open cover of $K$ then there are finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}.$$

# Compact sets - examples

## Example 1

Every finite subset of $\mathbb{R}$ is compact.

## Example 2

$K = \{\frac{1}{n} \ : \ n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$.

- Indeed, let $(V_\alpha)_{\alpha \in A}$ be an open cover of $K$, then there is $\alpha_0 \in A$ such that $0 \in V_{\alpha_0}$ since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $V_{\alpha_0}$ is open thus it contains all but finitely many $\frac{1}{n}$'s. In other words, there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\frac{1}{n} \in V_{\alpha_0}$. Then, for each $j \in \{1, 2, \ldots, n_0 - 1\}$ we can pick $\alpha_j \in A$ so that $\frac{1}{j} \in V_{\alpha_j}$ and we see

$$K \subseteq \bigcup_{j=0}^{n_0} V_{\alpha_j}.$$

# Theorem

## Theorem

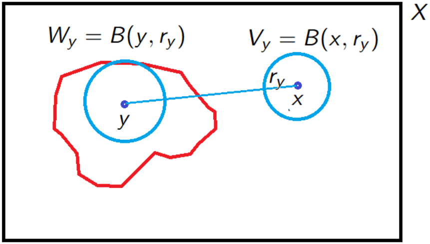
Compact subsets of metric spaces are closed.

**Proof.** Let $K$ be compact subset of a metric space $X$.

- We shall prove that $K^c$ is open in $X$. Let $x \in X \setminus K$. If $y \in K$, let

$$V_y = B(x, r_y) \quad \text{and} \quad W_y = B(y, r_y),$$

where $r_y < \frac{1}{2}\rho(x, y)$, then $V_y \cap W_y = \emptyset$.



# Proof

- Since $K$ is compact $K \subseteq \bigcup_{y \in K} W_y$, then we can find $y_1, \ldots, y_n \in K$ so that
$$K \subseteq \bigcup_{j=1}^{n} W_{y_j} = W.$$
- If $V = V_{y_1} \cap \ldots \cap V_{y_n}$ then $V$ is an open set containing $x$ and
$$V \cap W = \emptyset.$$
- Hence $x \in V \subseteq W^c \subseteq K^c$ thus $x$ is an interior point of $K^c$. □

# Theorem

## Theorem

Closed subsets of compact sets are compact.

**Proof.** Suppose that $F \subseteq K \subseteq X$ and $F$ is closed in $X$ and $K$ is compact.

- Let $(V_\alpha)_{\alpha \in A}$ be an open cover of $F$. Observe that

$$F \subseteq K \subseteq \underbrace{\bigcup_{\alpha \in A}}_{F \subseteq} V_\alpha \cup \underbrace{F^c}_{\text{open}}.$$

- The set $K$ is compact thus there is a finite subcover of

$$(V_\alpha)_{\alpha \in A} \cup \{F^c\}$$

  that covers $K$.
- But $F \subseteq K$ hence this is also a finite subcover of $F$ upon removing $F^c$ as desired. □

# Theorem

**Theorem**

If $(K_\alpha)_{\alpha \in A}$ is a collection of compact sets of a metric space $(X, \rho)$ such that the intersection of every finite subcollection of $(K_\alpha)_{\alpha \in A}$ is non-empty then

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

**Proof.** Fix a member $K_{\alpha_0}$ of $(K_\alpha)_{\alpha \in A}$ and set $G_\alpha = K_\alpha^c$.

- Suppose that

$$\bigcap_{\alpha \in A} K_\alpha = K_{\alpha_0} \cap \left( \bigcap_{\alpha \in A \setminus \{\alpha_0\}} K_\alpha \right) = \emptyset.$$

# Proof

- Then

$$K_{\alpha_0} \subseteq \bigcup_{\alpha \in A \setminus \{\alpha_0\}} G_\alpha.$$

- Since $K_{\alpha_0}$ is compact there are $\alpha_1, \ldots, \alpha_n \in A$ so that

$$K_{\alpha_0} \subseteq \bigcup_{j=1}^{n} G_{\alpha_j}.$$

- Hence

$$K_{\alpha_0} \cap K_{\alpha_1} \cap \ldots \cap K_{\alpha_n} = \emptyset,$$

which is a contradiction. So we must have

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

as desired. □

# Accumulation and isolated points

## Accumulation point

Let $(X, \rho)$ be a metric space, $x \in X$ is called **an accumulation point of** $E \subseteq X$ if for every open set $U \ni x$ we have

$$(E \setminus \{x\}) \cap U \neq \emptyset.$$

An accumulation point $x$ of $E \subseteq X$ is sometimes also called **a limit point of** $E$ or **a cluster point of** $E$.

## Isolated point

A point $x \in E$ is called **an isolated point of** $E$ if it is not an accumulation point of $E$.

# Perfect sets

## Perfect sets

We say that a subset $E$ of a metric space $(X, \rho)$ is **perfect** if $E$ is closed and every point of $E$ is its limit point or equivalently

$$E = \mathrm{acc}\ E.$$

## Theorem

Let $\emptyset \neq P \subseteq \mathbb{R}^k$ be a perfect set. Then $P$ is uncountable.

In the proof we will use a very useful proposition:

## Proposition

Every closed and bounded set of $\mathbb{R}^k$ is compact.

The proof of this proposition will be provided next time.

# Proof: 1/3

**Proof.** Since $P$ has limit points, $P$ must be infinite. In fact, for every $x \in P$ and $r > 0$

$$B(x, r) \cap P \quad \text{is infinite.}$$

- Suppose not, i.e. there is $x_0 \in P$ and $r_0 > 0$ such that

$$B(x_0, r_0) \cap P = \{x_1, \ldots, x_n\}.$$

- Consider

$$\rho(x_0, x_1), \ldots, \rho(x_0, x_n)$$

  and let

$$r = \min_{1 \leq i \leq n} \rho(x_0, x_i) > 0.$$

- Then

$$B(x_0, r) \cap P = \emptyset,$$

  thus $x_0$ is not a limit point, contradiction.

# Proof: 2/3

Now we can assume $\operatorname{card}(P) \geq \operatorname{card}(\mathbb{N})$. Suppose for a contradiction that $\operatorname{card}(P) = \operatorname{card}(\mathbb{N})$, i.e. $P = \{x_1, x_2, \ldots\}$.

- Let $V_1 = B(x_1, r)$, then of course $V_1 \cap P \neq \emptyset$. Suppose that $V_n$ has been constructed so that $V_n \cap P \neq \emptyset$.
- Since every point of $P$ is a limit point of $P$ there is an open set $V_{n+1}$ such that
  1. (i) $\operatorname{cl}(V_{n+1}) \subseteq V_n$,
  2. (ii) $x_{n+1} \notin \operatorname{cl}(V_{n+1})$,
  3. (iii) $V_{n+1} \cap P \neq \emptyset$.
- Let $K_n = \operatorname{cl}(V_n) \cap P$, this set is closed and bounded, thus compact. Since $x_n \notin K_{n+1}$, no point of $P$ lies in $\bigcap_{n=1}^{\infty} K_n$, but $K_n \subseteq P$, so

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

# Proof: 3/3

- On the other hand, $K_n \neq \emptyset$, compact, and $K_{n+1} \subseteq K_n$, and the family $K_n$ has a finite intesection property, i.e. any finite intersection of members of $(K_n)_{n\in\mathbb{N}}$ is nonempty,

$$K_{n_1} \cap \ldots \cap K_{n_k} \neq \emptyset.$$

- Thus

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset,$$

which is a contradiction. Hence $P$ must be uncountable. □

**Corollary**

Every interval $[a, b]$ with $a < b$, and also $\mathbb{R}$ are uncountable.