Lesson 18

Compact Sets, Connected Sets, and Cantor set

MATH 311, Section 4, FALL 2022

November 8, 2022

Totally bounded sets

Totally bounded set

Let (X, ρ) be a metric space, $E \subseteq X$ is called **totally bounded** if for every $\varepsilon > 0$, the set E can be covered by finitely many balls of radius ε.

- It means that there is $N_\varepsilon \in \mathbb{N}$ so that

$$E \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \quad \text{for some} \quad x_1, x_2, \ldots, x_{N_\varepsilon} \in X.$$

Remark 1

If E is totally bounded so is $\mathrm{cl}\,(E)$. Indeed,

$$E \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \Longrightarrow \mathrm{cl}\,(E) \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, 2\varepsilon).$$

Remark

Remark 2

Every totally bounded set E is bounded. If

$$x, y \in E \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon),$$

then say $x \in B(x_1, \varepsilon)$, $y \in B(x_2, \varepsilon)$ and

$$\begin{aligned} \rho(x, y) &\leq \rho(x, x_1) + \rho(x_1, x_2) + \rho(x_2, y) \\ &\leq \varepsilon + \max\{\rho(x_i, x_j) \ : \ 1 \leq i, j \leq N_\varepsilon\} + \varepsilon. \end{aligned}$$

- The converse is false in general.

Characterization of compactness

Theorem

If E is a subset of a metric space (X, ρ) the following are equivalent.

(a) E is complete and totally bounded.

(b) (The Bolzano–Weierstrass property) Every sequence in E has a subsequence that converges to a point of E.

(c) (The Heine–Borel property) If $(V_\alpha)_{\alpha \in A}$ is an open cover of E then there is finite $F \subseteq A$ such that $(V_\alpha)_{\alpha \in F}$ covers E.

Remark

This theorem can be thought of as a characterization of compactness in metric spaces.

Proof of (a) $\Rightarrow$ (b): 1/2

Suppose that (a) holds and $(x_n)_{n\in\mathbb{N}} \subseteq E$. We find $(x_{n_k})_{k\in\mathbb{N}}$ such that $\rho(x_{n_k}, x_0) \xrightarrow[k\to\infty]{} 0$ for some $x_0 \in E$.

- E can be covered by finitely many balls of radius $1/2$. At least one of them must contain x_n for infinitely many $n \in \mathbb{N}$:
  - say $x_n \in B_1$ for $n \in \mathbb{N}_1 \subseteq \mathbb{N}$ and $\operatorname{card}(\mathbb{N}_1) = \operatorname{card}(\mathbb{N})$.
- Now $E \cap B_1$ can be covered by finitely many balls of radius $1/4$. At least one of them must contain x_n for infinitely many $n \in \mathbb{N}$:
  - say $x_n \in B_2$ for $n \in \mathbb{N}_2 \subseteq \mathbb{N}_1$ and $\operatorname{card}(\mathbb{N}_2) = \operatorname{card}(\mathbb{N})$.
- Continuing inductively we obtain a sequence of balls B_j of radius 2^{-j} and decreasing sequence of subsets $\mathbb{N}_j$ of $\mathbb{N}$ such that
  - $x_n \in B_j$ for $n \in \mathbb{N}_j$, $\mathbb{N}_{j+1} \subseteq \mathbb{N}_j \subseteq \mathbb{N}$, $\operatorname{card}(\mathbb{N}_j) = \operatorname{card}(\mathbb{N})$.

Proof of (a) $\Rightarrow$ (b): 2/2

- Pick $n_1 \in \mathbb{N}_1$, $n_2 \in \mathbb{N}_2, \ldots$ such that

$$n_1 < n_2 < n_3 < \ldots.$$

- Then $(x_{n_j})_{j\in\mathbb{N}}$ is a Cauchy sequence for

$$\rho(x_{n_j}, x_{n_k}) < 2^{1-j} \quad \text{if} \quad k \geq j,$$

since $x_{n_j}, x_{n_k} \in B_j$ and

$$\mathrm{diam}\,(B_j) \leq 2^{1-j}.$$

- Since E is complete the sequence $(x_{n_k})_{k\in\mathbb{N}}$ has a limit in E and the implication (a) $\Rightarrow$ (b) is proved. □

Proof of (b) $\Rightarrow$ (a)

We show that of either condition in (a) fails then so does (b).

- If E is not complete there is a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$, with no limit in E. No subsequence of $(x_n)_{n\in\mathbb{N}}$ can converge in E, for otherwise the whole sequence would converge to the same limit.

- On the other hand if E is not totally bounded, let $\varepsilon > 0$ be such that E cannot be covered by finitely many balls of radius $\varepsilon > 0$. Choose $x_n \in E$ inductively as follows. Let $x_1 \in E$, and having chosen $x_1, \ldots, x_n$ pick

$$x_{n+1} \in E \setminus \bigcup_{j=1}^{n} B(x_j, \varepsilon),$$

then $\rho(x_n, x_m) \geq \varepsilon$ for all $n \neq m$, so $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence. Thus (b) $\Rightarrow$ (a). □

Proof of theorem (a) and (b) $\Rightarrow$ (c)

It suffices to show that if (b) holds and $(V_\alpha)_{\alpha \in A}$ is an open cover of E then the following claim holds:

Claim

There exists $\varepsilon > 0$ such that every ball of radius $\varepsilon > 0$ that intersects E is contained in some V_α.

Then E can be covered by finitely many such balls by (a) this allows us to find a finite subcover of $(V_\alpha)_{\alpha \in A}$.

Proof of Claim

Suppose for a contradiction that the claim is not true.

- For each $n \in \mathbb{N}$ there is a ball B_n of radius 2^{-n} such that $B_n \cap E \neq \emptyset$ and B_n is contained in no V_α.
- Pick $x_n \in B_n \cap E$. Using (b), (by passing to a subsequence if necessary) we may assume $\lim_{n\to\infty} \rho(x_n, x) = 0$ for some $x \in E$.
- We have $x \in V_\alpha$ for some $\alpha \in A$ and since V_α is open there is $\varepsilon > 0$ so that $B(x, \varepsilon) \subseteq V_\alpha$.
- If n is large enough so that $\rho(x_n, x) < \frac{\varepsilon}{3}$ and $2^{-n} < \frac{\varepsilon}{3}$, then $B_n \subseteq B(x, \varepsilon) \subseteq V_\alpha$, which is contradiction.
- Indeed, pick $y \in B_n$, then

$$\rho(y, x) \leq \rho(x_n, y) + \rho(x_n, x) < 2^{1-n} + \frac{\varepsilon}{3} \leq \varepsilon.$$

This completes the proof of the implication (a) and (b) $\Rightarrow$ (c). □

Proof of (c) $\Rightarrow$ (b)

- If $(x_n)_{n\in\mathbb{N}} \subseteq E$, with no convergent sequence, for each $x \in E$ there is a ball B_x centered at x that contains x_n for only finitely many n.
- Otherwise, some sequence would converge to x. Then

$$(B_x)_{x\in E}$$

is a cover of E by open sets with no finite subcover. □

Compactness in $\mathbb{R}$

Theorem

Every closed and bounded set of $\mathbb{R}$ is compact.

Proof. We deduce compactness by showing completeness and total boundedness.

- Since every closed subset of $\mathbb{R}$ is complete it suffices to show that bounded subsets of $\mathbb{R}$ are totally bounded.
- Since every bounded set is contained in some interval $[-R, R]$ it is enough to show that $[-R, R]$ is totally bounded.
- Given $\varepsilon > 0$ pick an integer $k > \frac{R}{\varepsilon}$ and express $[-R, R]$ as the union of k intervals of equal length. □

Compactness in $\mathbb{R}^n$

Theorem

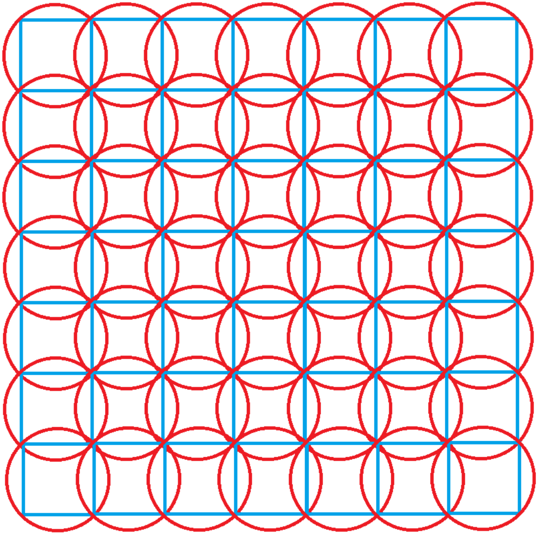
Every closed and bounded set of $\mathbb{R}^n$ is complete.

Proof. We deduce compactness by showing completeness and total boundedness.

- Since every closed subset of $\mathbb{R}^n$ is complete is suffices to show that bounded subsets of $\mathbb{R}^n$ are totally bounded.
- Since every bounded set is contained in some cube $Q = [-R, R]^n$ it is enough to show that Q is totally bounded.
- Given $\varepsilon > 0$ pick the integer $k > \frac{R\sqrt{n}}{\varepsilon}$ and express Q as the union of n^n congruent subcubes by dividing the interval $[-R, R]$ into k equal pieces.
- The side length of these subcubes is $\frac{2R}{k}$ and hence the diameter is $\sqrt{n}\left(\frac{2R}{k}\right) < 2\varepsilon$, so they are contained in the balls of radius ε about their centers.

□

$Q = [-R, R]^n$ is totally bounded



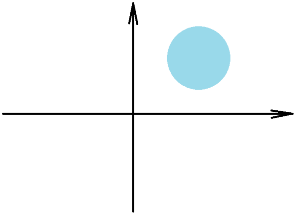
Example

Example

Determine if the set

$$X = \{(x, y) \in \mathbb{R}^2 \ : \ (x-1)^2 + (y-1)^2 < 1\}$$

is compact or not in $\mathbb{R}^2$ with Euclidean metric.



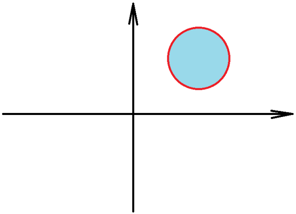
Solution. Note that $(2, 0)$ is an accumulation point of X, but $(2, 0) \notin X$. Therefore, X **is not closed**, so it is **not compact**. □

Example

Example

Determine if the set is compact or not in $\mathbb{R}^2$ with Euclidean metric:

$$X = \{(x, y) \in \mathbb{R}^2 \ : \ (x-1)^2 + (y-1)^2 \leq 1\}.$$



Solution. X contains all of its accumulation points so it is **closed**. It is contained in the ball $B(0, 10)$, so it is **bounded**. Therefore, by the previous theorem, it is **compact**. □

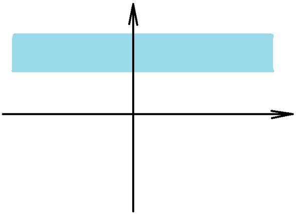
Example

Example

Determine if the set

$$X = \{(x, y) \in \mathbb{R}^2 \ : 1 < y < 2\}$$

is compact or not in $\mathbb{R}^2$ with Euclidean metric.



Solution. Note that $(0, 2)$ is an accumulation point of X, but $(0, 2) \notin X$. Therefore, X **is not closed**, so it is **not compact**. □

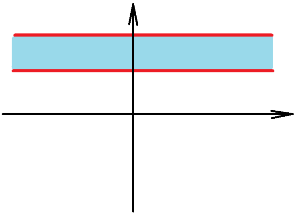
Example

Example

Determine if the set

$$X = \{(x, y) \in \mathbb{R}^2 \ : 1 \leq y \leq 2\}$$

is compact or not in $\mathbb{R}^2$ with Euclidean metric.



Solution. In can be checked that X is closed, although it is not contained in any ball, so it is **not bounded**, so it is **not compact**. □

Examples

Example

Determine if the set $\mathbb{Q}$ is compact in $\mathbb{R}$.

Solution. $\mathbb{Q}$ is not contained in any interval, so it is **not compact**. $\square$

Example

Determine if the set $\mathbb{Q} \cap [0, 1]$ is compact in $\mathbb{R}$.

Solution. $\mathbb{Q}$ is contained in $(-1, 2)$, but $\mathrm{cl}\, \mathbb{Q} \cap [0, 1] = [0, 1] \neq \mathbb{Q} \cap [0, 1]$, so it is not closed, so it is **not compact**. $\square$

Separated and connected sets

Separated sets

Two subsets A and B of a metric space (X, ρ) are said to be **separated** if both

$$A \cap \mathrm{cl}\,(B) = \emptyset \quad \text{and} \quad \mathrm{cl}\,(A) \cap B = \emptyset.$$

In other words, no points of A lies in the closure of B and vice versa.

Connected set

A set $E \subseteq X$ is said to be **connected** if E is not a union of two nonempty separated sets.

Example

- $[0, 1]$ and $(1, 2)$ are not separated since 1 is a limit point of $(1, 2)$.
- However, $(0, 1)$ and $(1, 2)$ are separated.

Theorem

Theorem

$E \subseteq \mathbb{R}$ is connected iff for all $x, y \in E$ if $x < z < y$, then $z \in E$.

Proof ($\Longrightarrow$). If there exist $x, y \in E$ and $z \in (x, y)$ such that $z \notin E$, then

$$E = A_z \cup B_z, \quad \text{where} \quad A_z = E \cap (-\infty, z) \quad \text{and} \quad B_z = E \cap (z, \infty).$$

Since $x \in A_z$ and $y \in B_z$, then $A_z \neq \emptyset$, $B_z \neq \emptyset$ and also $A_z \subseteq (-\infty, z)$, $B_z \subseteq (z, \infty)$, so they are separated. Hence E is not connected.

Proof

Proof ($\Longleftarrow$). Conversely, suppose that E is not connected.

- Then there are non-empty separated sets A, B such that $A \cup B = E$.
- Pick $x \in A$ and $y \in B$ and without loss of generality assume $x < y$. Define
$$z = \sup\left(A \cap [x, y]\right).$$
hence $z \in \mathrm{cl}\left(A\right)$ and $z \notin B$. In particular, $x \leq z < y$.
- If $z \notin A$ it follows $x < z < y$ and $z \notin E$.
- If $z \in A$ then $z \notin \mathrm{cl}\left(B\right)$ hence there is z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$. □

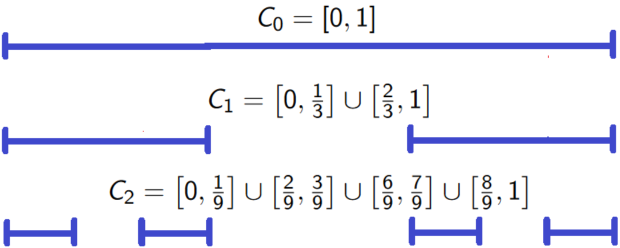
Example

Prove that $X = \mathbb{R} \setminus \{0\}$ is not connected.

Solution. We have $-1, 1 \in X$, but $-1 < 0 < 1$ and $0 \notin X$, so X is not connected. □

There exists a perfect set in $\mathbb{R}$ which contains no segment.

- Let $C_0 = [0, 1]$. Given C_n that consist of 2^n disjoint closed intervals each of length 3^{-n} take each of these intervals and delete the open middle third to produce two closed intervals each of length 3^{-n-1}.



- Take C_{n+1} to be the union of 2^{n+1} closed intervals so formed and continue.

Cantor set

Cantor set

The set

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

is called **the Cantor set** or ternary Cantor set.

- Each $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ is closed and bounded thus compact, and the family $(C_n)_{n\in\mathbb{N}}$ has finite intersection property thus the Cantor set is compact and $\mathcal{C} \neq \emptyset$.

Property (*)

By the construction for each $k, m \in \mathbb{N}$ we see that no segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \qquad \text{has a point in common with } \mathcal{C}.$$

Properties of the Cantor set

- Since every segment (α, β) contains a segment of the form (*) if m is sufficiently large, since the set

$$\left\{ \frac{\ell}{3^m} \;:\; m \in \mathbb{N} \text{ and } 0 \le \ell \le 3^m - 1 \right\}$$

is dense in $[0, 1]$. Thus $\mathcal{C}$ contains no segment (α, β). This also shows $\mathrm{int}\, \mathcal{C} = \emptyset$.

- To prove that $\mathcal{C}$ is perfect it is enough to show that $\mathcal{C}$ contains no isolated point. Let $x \in \mathcal{C}$ and let I_n be the unique interval from C_n which contains $x \in I_n$. Let x_n be the endpoint of I_n such that $x \neq x_n$. It follows from the construction of $\mathcal{C}$ that $x_n \in \mathcal{C}$. Hence x is a limit point of $\mathcal{C}$ thus $\mathcal{C}$ is perfect.

More about Cantor set

- Each component of C_n can be described as the set

$$C_n = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_j}{3^j} \ : \ \varepsilon_j \in \{0,1,2\} \text{ and } \varepsilon_j \neq 1 \text{ for } 1 \leq j \leq n \right\}.$$

- Consequently,

$$\mathcal{C} = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_j}{3^j} \ : \ \varepsilon_j \in \{0,2\} \right\}.$$

Fact

Fact

Any number $\sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j}$ is uniquely determined by its sequence $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ with $\varepsilon_j \in \{0, 2\}$.

Proof. Take $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$, $\delta = (\delta_j)_{j\in\mathbb{N}}$ with $\varepsilon_j, \delta_j \in \{0, 2\}$ such that $\varepsilon \neq \delta$. Let $N = \min\{j \in \mathbb{N} \ : \ \varepsilon_j \neq \delta_j\}$ and assume $0 = \varepsilon_N < \delta_N = 2$. Then

$$\sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = \sum_{j=1}^{N-1} \frac{\varepsilon_j}{3^j} + \sum_{j=N+1}^{\infty} \frac{\varepsilon_j}{3^j} \le \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^{N+1}} \sum_{j=0}^{\infty} \frac{1}{3^j}$$

$$\le \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^{N+1}} \underbrace{\frac{1}{1-\frac{1}{3}}}_{\frac{3}{2}} = \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{1}{3^N} < \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^N} \le \sum_{j=1}^{\infty} \frac{\delta_j}{3^j}.$$

This completes the proof. □

Remarks

Remark

We have two different representations

$$\frac{1}{3} = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = A, \quad \varepsilon_1 = 1, \quad \varepsilon_j = 0 \quad \text{for} \quad j \geq 2.$$

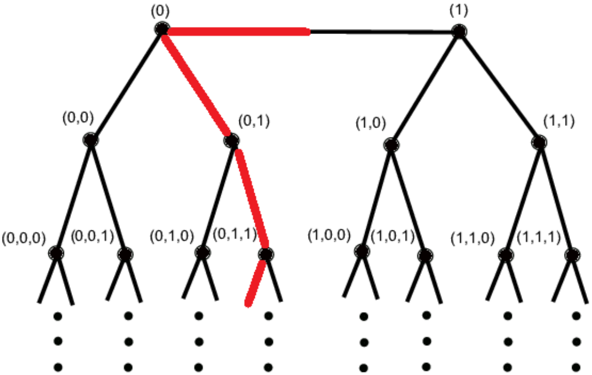
$$\frac{1}{3} = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = B, \quad \varepsilon_1 = 0, \quad \varepsilon_j = 2 \quad \text{for} \quad j \geq 2.$$

There is a bijection $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{C}$ defined by

$$\phi(z) = \frac{2}{3} \sum_{j=0}^{\infty} \frac{z_j}{3^j} \quad \text{for} \quad z = (z_j)_{j\in\mathbb{N}}, \quad z_j \in \{0,1\},$$

and consequently $\mathrm{card}\,(\mathcal{C}) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}) = \mathrm{card}\,(\mathbb{R}) = \mathfrak{c}$.

Cantor tree



$$\varepsilon = (0, 1, 1, 0, \varepsilon_4, \varepsilon_5, \dots)$$