Lesson 19

Continuous functions,
Continuous functions on compact sets

MATH 311, Section 4, FALL 2022

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Limits

Limits

Suppose $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ and p is a limit point of E. We write

$$f(x) \underset{x \to p}{\longrightarrow} q \quad \text{or} \quad \lim_{x \to p} f(x) = q.$$

if there is a point $q \in \mathbb{R}$ satisfying the following ε-δ condition:

- For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - q| < \varepsilon$$

  for all points $x \in E$ for which $0 < |x - p| < \delta$.

Theorem

Theorem (Characterizations of Continuity)

Let $E \subseteq \mathbb{R}$, $f : E \to \mathbb{R}$, and $p \in \mathbb{R}$ be as in the previous definition. Then

- **A** $\lim_{x\to p} f(x) = q$ iff
- **B** $\lim_{n\to\infty} f(p_n) = q$ for every sequence $(p_n)_{n\in\mathbb{N}}$ in E such that $p_n \neq p$ and $\lim_{n\to\infty} p_n = p$.

Proof (A)$\Longrightarrow$(B). Suppose that (A) holds. Choose $(p_n)_{n\in\mathbb{N}}$ like in condition (B). Let $\varepsilon > 0$ be given, then there exists $\delta > 0$ such that

$$|f(x) - q| < \varepsilon \quad \text{if} \quad x \in E \quad \text{and} \quad 0 < |x - p| < \delta.$$

Also there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $0 < |p_n - p| < \delta$. Thus we also have $|f(p_n) - q| < \varepsilon$ for $n \geq N$ showing that (B) holds. □

Proof (B)$\Longrightarrow$(A)

Proof (B)$\Longrightarrow$(A). Conversely suppose (A) is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ) for which

$$|f(x) - q| \geq \varepsilon \quad \text{but} \quad 0 < |x - p| < \delta.$$

Taking $\delta_n = \frac{1}{n}$ for each $n \in \mathbb{N}$ we thus find a sequence $(p_n)_{n\in\mathbb{N}}$ in E satisfying $\lim_{n\to\infty} p_n = p$ but

$$|f(p_n) - q| \geq \varepsilon.$$

thus (B) is false as desired. □

Remark

It was possible to choose the sequence $(p_n)_{n\in\mathbb{N}}$ in E in one step thanks to the Axiom of Choice. Without assuming the Axiom of Choice the previous theorem is not provable.

Theorem

Theorem

Suppose that $E \subseteq \mathbb{R}$, and p is a limit point of E. Let $f, g : E \to \mathbb{R}$ be functions such that

$$\lim_{x \to p} f(x) = A \quad \text{and} \quad \lim_{x \to p} g(x) = B.$$

Then

a. $\lim_{x \to p}(f + g)(x) = A + B$,
b. $\lim_{x \to p}(f \cdot g)(x) = A \cdot B$,
c. $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$ and $g(x) \neq 0$ for $x \in E$.

Continuous function

Continuous at the point p

Suppose that $E \subseteq X$, $p \in E$ and $f : E \to \mathbb{R}$. The function f is said to be **continuous at point** p if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(p)| < \varepsilon$$

for all points $x \in E$ for which

$$|x - p| < \delta.$$

Continuous function

If the function $f : E \to \mathbb{R}$ is continuous at every point of E then f is said to be **continuous** on E.

Example

Example

Let us define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Determine if f is continuous or not at the point 0.

Solution. Let us consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \sqrt{2}/n$. Then $\lim_{n\to\infty} a_n = 0$ and $a_n \notin \mathbb{Q}$, so $f(a_n) = 0$. Then

$$\lim_{n\to\infty} f(a_n) = 0 \neq 1 = f(0),$$

so f is **not continuous at point** 0. □

Remark

Remark

If p is an isolated point of E then our definition implies that every function f which has E as its domain is continuous at p. For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $e \in E$ for which

$$|x - p| < \delta$$

is $x = p$, then

$$|f(x) - f(p)| = 0 < \varepsilon.$$

Fact

In the situation of the definition of continuity assume also that p is a limit point of E. Then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Proof. It is obvious if we compare two previous definitions. □

Theorem

Theorem

Suppose that $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ and $g : f[E] \to \mathbb{R}$ be given and define $h : E \to \mathbb{R}$ by

$$h(x) = g(f(x)), \quad x \in E.$$

If f is continuous at a point $p \in E$ and g is continuous at the point $f(p)$, then h is continuous at p. In other words

$$\lim_{x \to p} h(x) = \lim_{x \to p} g(f(x)) = g(f(p)) = h(p).$$

Proof

Let $\varepsilon > 0$ be given.

- Since g is continuous at $f(p)$ there is $\eta > 0$ such that

$$|g(y) - g(f(p))| < \varepsilon \quad \text{if} \quad |y - f(p)| < \eta \quad \text{and} \quad y \in f[E].$$

- Since f is continuous at p, there is $\delta > 0$ such that

$$|f(x) - f(p)| < \eta \quad \text{if} \quad |x - p| < \delta \quad \text{and} \quad x \in E.$$

- If follows that

$$|h(x) - h(p)| = |g(f(x)) - g(f(p))| < \varepsilon$$

if $|x - p| < \delta$ and $x \in E$. Thus h is continuous at $p \in E$. □

Example

Example

Assume that $f : \mathbb{R} \to (0, \infty)$ is continuous for all $x \in \mathbb{R}$. Prove that $h(x) = \sqrt{f(x)}$ is continuous.

Solution. Let us note that the function $g : (0, \infty) \to (0, \infty)$ defined by

$$g(x) = \sqrt{x}$$

is continuous. We have

$$h = g \circ f,$$

so h is continuous by the previous theorem. □

Theorem

Theorem

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ iff $f^{-1}[V]$ is open in $\mathbb{R}$ for every open set V in $\mathbb{R}$.

Proof. Suppose that f is continuous on $\mathbb{R}$ and $V \subseteq \mathbb{R}$ is open.

- We have to show that $f^{-1}[V]$ is open in $\mathbb{R}$. Let $p \in f^{-1}[V]$. Since V is open $(f(p) - \varepsilon, f(p) + \varepsilon) \subseteq V$ for some $\varepsilon > 0$.
- Since f is continuous at $p \in X$ there is $\delta > 0$ such that

$$|f(x) - f(p)| < \varepsilon \quad \text{if} \quad |x - p| < \delta.$$

Thus

$$(p - \delta, p + \delta) \subseteq f^{-1}[V] = \{x \in \mathbb{R} \ : \ f(x) \in V\}.$$

Proof

Conversely, suppose $f^{-1}[V]$ is open in X for any open $V \subseteq \mathbb{R}$.

- Fix $p \in X$ and $\varepsilon > 0$ and consider
$$V = (f(p) - \varepsilon, f(p) + \varepsilon)$$
which is open thus $f^{-1}[V]$ is open, hence there is $\delta > 0$ so that $(p - \delta, p + \delta) \subseteq f^{-1}[V]$.
- Thus if $|x - p| < \delta$, then $x \in f^{-1}[V]$, hence
$$f(x) \in V = (f(p) - \varepsilon, f(p) + \varepsilon) \iff |f(x) - f(p)| < \varepsilon.$$
□

Corollary

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff $f^{-1}[C]$ is closed in $\mathbb{R}$ for any closed set C in $\mathbb{R}$.

Proof. A set is closed iff its complement is open. We are done by invoking the previous theorem, since $f^{-1}[E^c] = (f^{-1}[E])^c$ for every open set $E \subseteq \mathbb{R}$. □

Example

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $a \in R$. Prove that the set

$$A = \{x \in \mathbb{R} \ : \ f(x) > a\}$$

is open.

Solution: We have

$$\{x \in \mathbb{R} \ : \ f(x) > a\} = f^{-1}[(a, \infty)]$$

and (a, ∞) is open in $\mathbb{R}$, so by the previous theorem, A is open. □

Theorem

Theorem

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Then $f+g$, $f \cdot g$, and $\frac{f}{g}$ are continuous. In the last case we assume $g(x) \neq 0$ for all $x \in \mathbb{R}$.

Example 1

Every polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is a continuous function on $\mathbb{R}$.

Example 2

The exponential function $f(x) = e^x$ is continuous as we have shown that for any $(a_n)_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} a_n = a$ one has $\lim_{n\to\infty} e^{a_n} = e^a$.

Examples

Example 3

$f(x) = |x|$ is continuous on $\mathbb{R}$ since $|f(x) - f(y)| \leq |x - y|$.

Example 4

$f(x) = \lfloor x \rfloor = \max\{n \in \mathbb{Z} \ : \ n \leq x\}$ is **NOT** continuous at any $x \in \mathbb{Z}$.

Example 5

$f(x) = x^\alpha$ for any $\alpha \in \mathbb{R}$ is continuous on $(0, \infty)$.

Example 6

If $f, g : X \to \mathbb{R}$ are continuous then $\max\{f, g\}$ and $\min\{f, g\}$ are continuous as well. Indeed,

$$\max\{f, g\} = \frac{f + g + |f - g|}{2}, \qquad \min\{f, g\} = \frac{f + g - |f - g|}{2}.$$

Continuity and compactness

Bounded function

A mapping $f : E \to \mathbb{R}$ is said to be **bounded** if there is a number $M > 0$ such that

$$|f(x)| \leq M \quad \text{for all} \quad x \in E.$$

Theorem (4.4.1)

Suppose that $f : X \to \mathbb{R}$ is a continuous function and $X \subseteq \mathbb{R}$ is compact. Then $f[X]$ is compact in $\mathbb{R}$.

Proof

Let $(V_\alpha)_{\alpha \in A}$ be an open cover of $f[X]$, i.e.

$$f[X] \subseteq \bigcup_{\alpha \in A} V_\alpha.$$

Since f is continuous then each set $f^{-1}[V_\alpha]$ is open in X. Since X is compact and

$$X \subseteq \bigcup_{\alpha \in A} f^{-1}[V_\alpha]$$

thus there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ so that

$$X \subseteq \bigcup_{j=1}^{n} f^{-1}[V_{\alpha_j}].$$

Since $f[f^{-1}[E]] \subseteq E$ we have

$$f[X] \subseteq f\Big[\bigcup_{j=1}^{n} f^{-1}[V_{\alpha_j}]\Big] \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}. \qquad \square$$

Corollary

Corollary

If $f : X \to \mathbb{R}$ is continuous on a compact set $X \subseteq \mathbb{R}$ then $f[X]$ is closed and bounded in $\mathbb{R}$. Specifically, f is bounded.

Theorem

Suppose $f : X \to \mathbb{R}$ is continuous on a compact set $X \subseteq \mathbb{R}$ and

$$M = \sup_{p \in X} f(p) \quad \text{and} \quad m = \inf_{p \in X} f(p).$$

Then there are $p, q \in X$ such that

$$f(p) = M \quad \text{and} \quad f(q) = m.$$

Proof. $f[X] \subseteq \mathbb{R}$ is closed and bounded. Thus M and m are members of $f[X]$ and we are done. □

Theorem

Theorem

Suppose f is continuous injective mapping of a compact set $X \subseteq \mathbb{R}$ onto a set $Y \subseteq \mathbb{R}$. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x, \quad x \in X$$

is a continuous mapping of Y onto X.

Proof. The inverse $f^{-1} : Y \to X$ is well defined since $f : X \to Y$ is one-to-one and onto. It suffices to prove that $f[V]$ is open in Y for every open set V in X. Fix $V \subseteq X$ open, V^c is closed in X thus compact, hence $f[V^c]$ is compact subset of Y and consequently $f[V^c]$ is closed. Since $f : X \to Y$ is one-to-one and onto, hence

$$f[V] = (f[V^c])^c$$

and, consequently, $f[V]$ is open as desired. □