Lesson 20

Continuity, compactness and connectivity, Uniform continuity, Sets of Discontinuity

MATH 311, Section 4, FALL 2022

November 15, 2022

Continuity and compactness

Bounded function

A mapping $f : E \to \mathbb{R}$ is said to be **bounded** if there is a number $M > 0$ such that

$$|f(x)| \leq M \quad \text{for all} \quad x \in E.$$

Theorem

Suppose that $f : X \to \mathbb{R}$ is a continuous function and $X \subseteq \mathbb{R}$ is compact. Then $f[X]$ is compact in $\mathbb{R}$.

Proof

Let $(V_\alpha)_{\alpha\in A}$ be an open cover of $f[X]$, i.e.

$$f[X] \subseteq \bigcup_{\alpha\in A} V_\alpha.$$

Since f is continuous then each set $f^{-1}[V_\alpha]$ is open in X. Since X is compact and

$$X \subseteq \bigcup_{\alpha\in A} f^{-1}[V_\alpha]$$

thus there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ so that

$$X \subseteq \bigcup_{j=1}^{n} f^{-1}[V_{\alpha_j}].$$

Since $f[f^{-1}[E]] \subseteq E$ we have

$$f[X] \subseteq f\Big[\bigcup_{j=1}^{n} f^{-1}[V_{\alpha_j}]\Big] \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}. \qquad \square$$

Corollary

Corollary

If $f : X \to \mathbb{R}$ is continuous on a compact set $X \subseteq \mathbb{R}$ then $f[X]$ is closed and bounded in $\mathbb{R}$. Specifically, f is bounded.

Theorem

Suppose $f : X \to \mathbb{R}$ is continuous on a compact set $X \subseteq \mathbb{R}$ and

$$M = \sup_{p \in X} f(p) \quad \text{and} \quad m = \inf_{p \in X} f(p).$$

Then there are $p, q \in X$ such that

$$f(p) = M \quad \text{and} \quad f(q) = m.$$

Proof. $f[X] \subseteq \mathbb{R}$ is closed and bounded. Thus M and m are members of $f[X]$ and we are done. □

Theorem

Theorem

Suppose f is continuous injective mapping of a compact set $X \subseteq \mathbb{R}$ onto a set $Y \subseteq \mathbb{R}$. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x, \quad x \in X$$

is a continuous mapping of Y onto X.

Proof. The inverse $f^{-1} : Y \to X$ is well defined since $f : X \to Y$ is one-to-one and onto. It suffices to prove that $f[V]$ is open in Y for every open set V in X. Fix $V \subseteq X$ open, V^c is closed in X thus compact, hence $f[V^c]$ is compact subset of Y and consequently $f[V^c]$ is closed. Since $f : X \to Y$ is one-to-one and onto, hence

$$f[V] = (f[V^c])^c$$

and, consequently, $f[V]$ is open as desired. □

Continuity and connectivity

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and if E is a connected subset of $\mathbb{R}$ then $f[E]$ is connected in $\mathbb{R}$.

Proof. Assume for a contradiction that $f[E] = A \cup B$, where A and B are nonempty separated sets in $\mathbb{R}$. Put

$$G = E \cap f^{-1}[A] \quad \text{and} \quad H = E \cap f^{-1}[B].$$

Then $E = G \cup H$ and neither G nor H is empty.

- Since $A \subseteq \mathrm{cl}\,(A)$ we have $G \subseteq f^{-1}[\mathrm{cl}\,(A)]$ and the latter set is closed since f is continuous hence $\mathrm{cl}\,(G) \subseteq f^{-1}[\mathrm{cl}\,(A)]$.
- Hence
$$f[\mathrm{cl}\,(G)] \subseteq f[f^{-1}[\mathrm{cl}\,(A)]] \subseteq \mathrm{cl}\,(A).$$

Proof

- Since $f[H] \subseteq B$ and $\mathrm{cl}\,(A) \cap B = \emptyset$ we conclude that

$$f[H \cap \mathrm{cl}\,(G)] \subseteq f[\mathrm{cl}\,(G)] \cap f[H] \subseteq \mathrm{cl}\,(A) \cap B = \emptyset,$$

  so $H \cap \mathrm{cl}\,(G) = \emptyset$.
- The same argument shows that $\mathrm{cl}\,(G) \cap H = \emptyset$.
- Thus G and H are separated sets, which is **a contradiction since E is connected**. □

Darboux property

Darboux property (intermediate value theorem)

Let f be a continuous function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there is a point $x \in (a, b)$ such that

$$f(x) = c.$$

A similar result holds if $f(a) > f(b)$.

Proof. $[a, b]$ is connected so $f\big[[a, b]\big]$ is connected in $\mathbb{R}$ as well by the previous theorem. Thus if $f(a) < c < f(b)$, then $c \in f\big[[a, b]\big]$, so there is $x \in [a, b]$ so that $f(x) = c$. □

Remark

The theorem stated above is sometimes called **Darboux property** or **the intermediate value theorem**.

Example

Exercise

Prove that the equation

$$x^3 - x^2 + 2x + 3 = 0$$

has a solution x_0 such that $-1 \leq x_0 \leq 0$.

Solution. Consider a continuous function

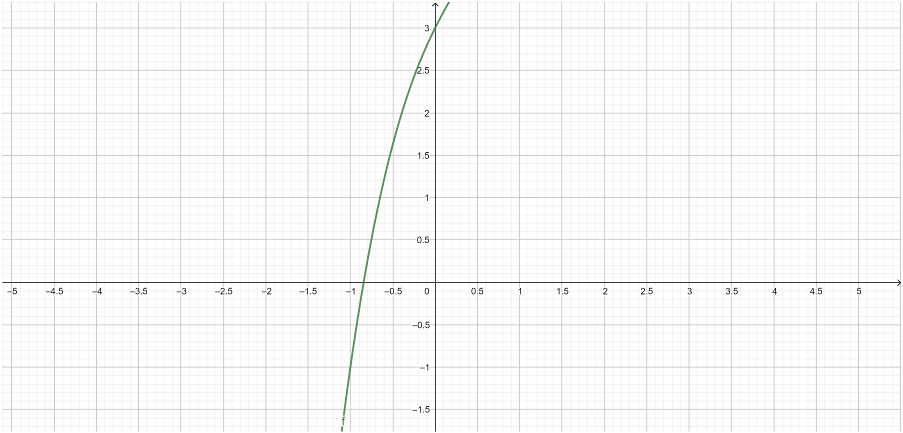
$$f(x) = x^3 - x^2 + 2x.$$

We calculate

$$f(-1) = -1, \quad \text{and} \quad f(0) = 3.$$

It follows by the Darboux property that there is $c \in [-1, 0]$ such that $f(c) = 0$. Thus c is a solution of our equation as desired. □

$f(x) = x^3 - x^2 + 2x + 3,\ x_0 \approx -0.8437$



Example

Exercise

Prove that the equation

$$x^3 = 20 + \sqrt{x}$$

has solution x_0.

Solution. Consider a continuous function

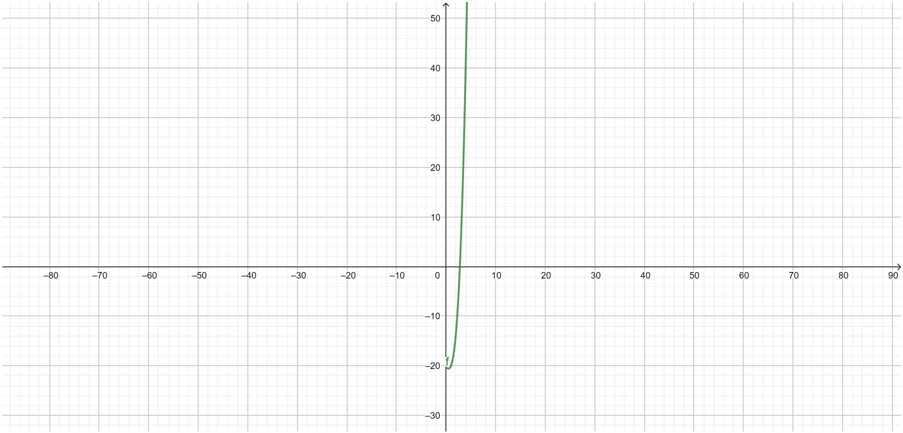
$$f(x) = x^3 - \sqrt{x} - 20.$$

We calculate

$$f(1) = -20 < 0, \quad \text{and} \quad f(4) = 42 > 0.$$

It follows by the Darboux property that there is $c \in [1, 4]$ such that $f(c) = 0$. Thus c is a solution of our equation as desired. □

$f(x) = x^3 - \sqrt{x} - 20,\ x_0 \approx 2,7879$



Uniformly continuous mappings

Uniformly continuous mappings

We say that $f : X \to \mathbb{R}$ is **uniformly continuous on** $X \subseteq \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

for all $x, y \in X$ for which

$$|x - y| < \delta.$$

Remark

- Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.

Remarks

Remark 1

- If f is continuous on X then for each $\varepsilon > 0$ and $p \in X$ there is $\delta > 0$ such that $|x - p| < \delta$ implies $|f(x) - f(p)| < \varepsilon$.
- Thus $\delta > 0$ **depends on** $p \in X$ **and** $\varepsilon > 0$.

Remark 2

- If f is uniformly continuous on X then for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ if $|x - y| < \delta$ then $|f(x) - f(p)| < \varepsilon$.
- Thus $\delta > 0$ **depends only on** $\varepsilon > 0$**, but is uniform for all** $x, y \in X$.

Remark 3

- **Uniform continuity implies continuity.**

Continuity on compact spaces becomes uniform

Theorem

Let $f : X \to \mathbb{R}$ be a continuous function defined on a compact set $X \subseteq \mathbb{R}$. Then f is uniformly continuous on X.

Proof. Let $\varepsilon > 0$ be given.

- Since f is continuous we can associate to each point $p \in X$ a positive number $\delta_p > 0$ such that if $|p - q| < \delta_p$, then $|f(p) - f(q)| < \frac{\varepsilon}{2}$.
- Observe that
$$X \subseteq \bigcup_{p \in X} \left(p - \frac{\delta_p}{2}, p + \frac{\delta_p}{2} \right).$$
- Since X is **compact** there are $p_1, p_2, \ldots, p_n \in X$ so that
$$X \subseteq \bigcup_{k=1}^{n} \left(p_k - \frac{\delta_{p_k}}{2}, p_k + \frac{\delta_{p_k}}{2} \right).$$

Proof

- Set
$$\delta = \frac{1}{2}\min\left(\delta_{p_1}, \dots, \delta_{p_n}\right) > 0.$$
- Let $p, q \in X$ be such that $|p - q| < \delta$, then there is $1 \le m \le n$ such that $p \in \left(p_m - \frac{\delta_{p_m}}{2}, p_m + \frac{\delta_{p_m}}{2}\right)$. Hence
$$|p_m - q| \le |q - p| + |p_m - p| \le \delta + \frac{\delta_{p_m}}{2} < \delta_{p_m}.$$
- Thus we conclude
$$|f(p) - f(q)| \le |f(p) - f(p_m)| + |f(p_m) - f(q)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. □

Example

Exercise

Let $f(x) = \frac{1}{\sqrt{x}}$. Determine if it is uniformly continuous on $[1, 2]$.

Solution. The interval $[1, 2]$ is compact and the function f is continuous at every point of $[1, 2]$. Hence, by the previous theorem, it is **uniformly continuous**. □

Exercise

Let

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

Determine if it is uniformly continuous on $[1, 2]$.

Solution. The interval $[1, 2]$ is compact and the function f is continuous at every point of $[1, 2]$. Hence, by the previous theorem, it is **uniformly continuous**. □

Example

Exercise

Let

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

Determine if it is uniformly continuous on $[0, 1]$.

Solution. Let us consider $a_n = \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 0$, but

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} n \neq f(0) = 0,$$

so f is not continuous at the point 0, so it is **not uniformly continuous**. □

Example

Exercise

Show that the function

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0, \end{cases} \qquad \text{is not uniformly continuous on } (0,1).$$

Solution. It can be checked that f is continuous on $(0,1)$.

- Suppose that f is uniformly continuous, then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in (0,1)$ if $|x-y| < \delta$ then
$$|f(x) - f(y)| < \varepsilon.$$
- We will use this condition with $\varepsilon = 1$ and $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$.
- This leads to a contradiction, since if $\frac{1}{n} < \delta$, then we see that

$$|x-y| = \frac{1}{n(n+1)} < \delta \quad \text{implies} \quad 1 = |n-n+1| = |f(x)-f(y)| < 1.$$

Discontinuities

Discontinuities

If x is a point in the domain of a function f at which f is not continuous we say that

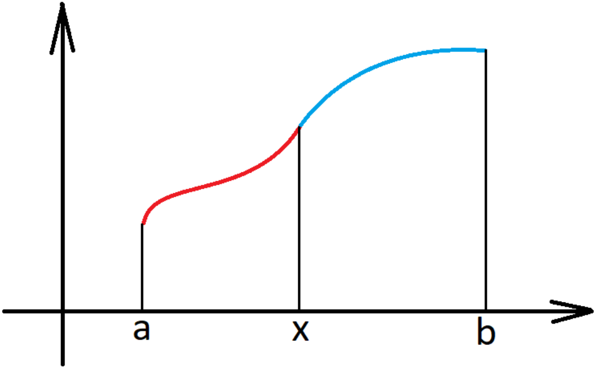
- f **is discontinuous** on X,
- or f **has a discontinuity at** $x \in X$.

Definition

Let $f : (a, b) \to \mathbb{R}$. Consider any x such that $a < x < b$.

- We write $f(x+) = q$ if $f(t_n) \xrightarrow[n\to\infty]{} q$ for all sequences $(t_n)_{n\in\mathbb{N}}$ in (x, b) such that $t_n \xrightarrow[n\to\infty]{} x$.
- Similarly, $f(x-) = q$ if $f(t_n) \xrightarrow[n\to\infty]{} q$ for all sequences $(t_n)_{n\in\mathbb{N}}$ in (a, x) such that $t_n \xrightarrow[n\to\infty]{} x$.
- It is clear that for any $x \in (a, b)$ $\lim_{t\to x} f(t)$ exists iff $f(x+) = f(x-) = \lim_{t\to x} f(t)$.

$f(x+)$ and $f(x-)$ - picture



Discontinuity of first and second kind

Let $f : (a, b) \to \mathbb{R}$ be given.

Discontinuity of the first kind

If f is discontinuous at a point x and if $f(x+)$ and $f(x-)$ exist, then f is said to have **discontinuity of the first kind** or **simple discontinuity at** x.

Discontinuity of the second kind

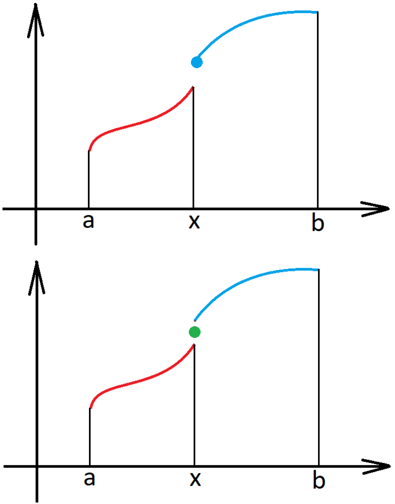
Otherwise the discontinuity is said to be **of the second type**.

Remark

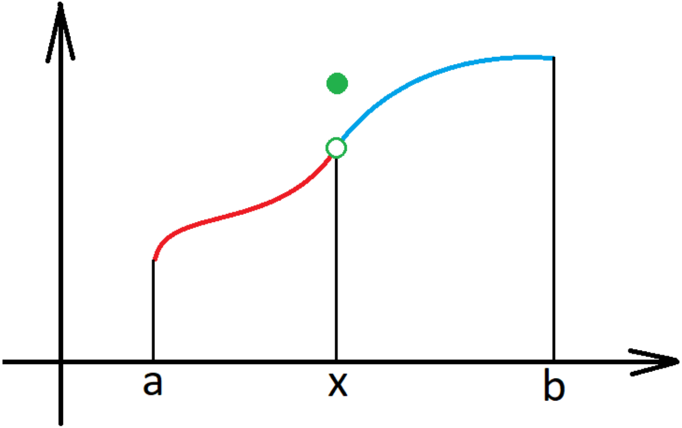
There are two ways in which a function can have a simple discontinuity:

- either $f(x+) \neq f(x-)$,
- or $f(x+) = f(x-) \neq f(x)$.

$f(x+) \neq f(x-)$



$f(x+) = f(x-) \neq f(x)$



Continuous from the left and from the right

Continuous from the left

If $f(x-) = f(x)$ for all $x \in (a, b)$ then we say that f is **continuous from the left**.

Continuous from the right

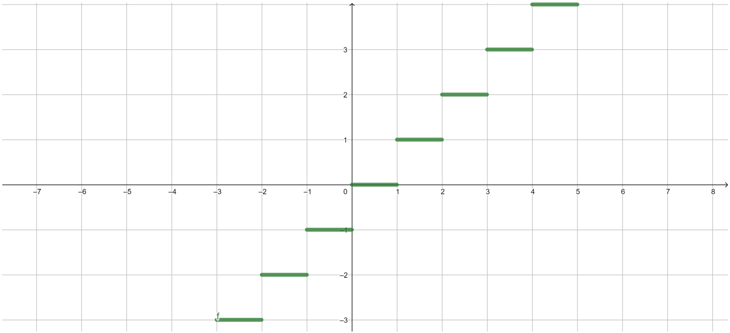
If $f(x+) = f(x)$ for all $x \in (a, b)$ then we say that f is **continuous from the right**.

Integer part

Integer part

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$$

is continuous from the right.

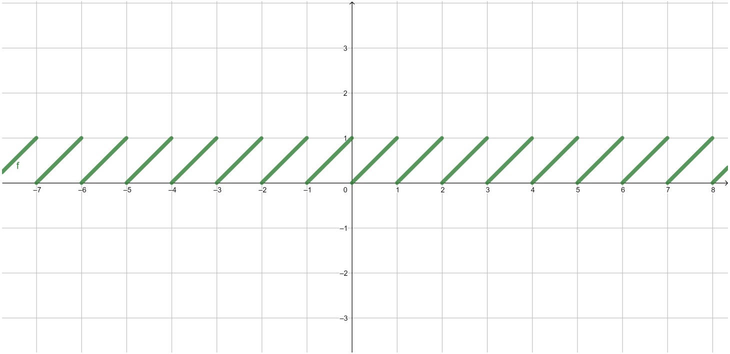


Fractional part

Fractional part

$$\{x\} = x - \lfloor x \rfloor$$

is also continuous from the right.



Examples involving characteristic function of $\mathbb{Q}$

Characteristic function of $\mathbb{Q}$

The function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

has a discontinuity of the second kind at every point x since neither $f(x+)$ nor $f(x-)$ exists.

Characteristic function of $\mathbb{Q}$ times linear functio

Define

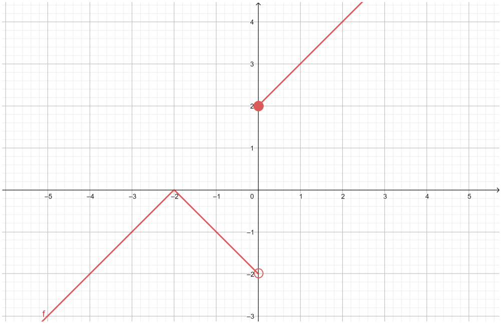
$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then f is continuous at $x = 0$, and f has a discontinuity of the second kind at every other point x since neither $f(x+)$ nor $f(x-)$ exists.

Example

An example of a function with a simple discontinuity at $x = 0$ that is continuous at every other point is given by the following formula

$$f(x) = \begin{cases} x + 2 & \text{if } x < -2, \\ -x - 2 & \text{if } x \in [-2, 0), \\ x + 2 & \text{if } x > 0. \end{cases}$$



Monotonically increasing and decreasing functions

Monotonically increasing (and decreasing) function

Let $f : (a, b) \to \mathbb{R}$, then f is said to be **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If $f(x) \geq f(y)$ we obtain the definition of **a monotonically decreasing function**.

Theorem

Let f be a monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point at $x \in (a, b)$. More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) \leq \inf_{x<t<b} f(t).$$

Furthermore, if $a < x < y < b$ then $f(x+) \leq f(y-)$. Analogous result remains true for monotonically decreasing functions.

Proof: 1/2

- The set
$$E = \{f(t)\ :\ a < t < x\}$$
is bounded by $f(x)$ hence $A = \sup E \in \mathbb{R}$ and $A \leq f(x)$.
- We have to show $f(-x) = A$.
- Let $\varepsilon > 0$ be given. Since $A = \sup E$ there is $\delta > 0$ such that $a < x - \delta < x$ and $A - \varepsilon < f(x - \delta) \leq A$. Since f is monotonic
$$f(x - \delta) \leq f(t) \leq A \quad \text{for} \quad t \in (x - \delta, x).$$
- Thus $A - \varepsilon < f(t) \leq A$, so
$$|f(x) - A| < \varepsilon \quad \text{for} \quad t \in (x - \delta, x).$$
- Thus $A = f(x-)$. In a similar way we prove $f(x+) = \inf_{x<t<b} f(t)$.

Proof: 2/2

- Next if $a < x < y < b$, then

$$f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t).$$

- Similarly

$$f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t).$$

- Thus

$$f(x+) = \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = f(y-). \qquad \square$$

Corollary

Monotonic functions have no discontinuities of the second kind.

Theorem

Theorem

Let $f : (a, b) \to \mathbb{R}$ be monotonic. Then the set of points of (a, b) of which f is discontinuous is at most countable.

Proof. Wlog we may assume that f is increasing.

- Let E be the set of points at which f is discontinuous.
- With every point $x \in E$ we associate a rational number $r(x) \in \mathbb{Q}$ such that

$$f(x-) < r(x) < f(x+),$$

  so $r : E \to \mathbb{Q}$.
- Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$ we see that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.
- We have established that the function $r : E \to \mathbb{Q}$ is injective, thus

$$\text{card}\,(E) \leq \text{card}\,(\mathbb{Q}) = \text{card}\,(\mathbb{N}). \qquad \square$$

Proof - illustration