Lesson 21

Derivatives, the Mean-Value Theorem and its Consequences
Higher Order Derivatives
Convex and Concave functions

MATH 311, Section 4, FALL 2022

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Derivative

Derivative

Let $f : [a, b] \to \mathbb{R}$. For any $x \in [a, b]$ form the quotient function

$$\phi(t) = \frac{f(t) - f(x)}{t - x}, \quad a < t < b, \quad t \neq x; \qquad \text{and define}$$

$$f'(x) = \lim_{t \to x} \phi(t)$$

provided the limit exists. We thus associate with the function f a function f' whose domain is the set of points x for which the limit $\lim_{t \to x} \phi(t)$ exists. The function f' is called **the derivative** of f.

Differentable function

- If f' is defined at point x, we say that f **is differentiable at** x.
- If f' is defined at every point of a set $E \subseteq [a, b]$ we say that f **is differentiable on** E.

Remarks – endpoints

Right-hand and left-hand limits

- It is possible to consider right-hand and left-hand limits of $\phi(t)$.
- This leads to the definition of right-hand and left-hand derivatives.
- In particular, at the endpoints a, b the derivative exists if exists a right-hand and left-hand derivative respectively.

Endpoints

- If f is defined on a segment (a, b) and if $a < x < b$, then $f'(x)$ is defined by

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x},$$

but $f'(a)$ and $f'(b)$ are not defined in this case.

Example

Exercise 1

Using the definition, calculate the derivative of $f(x) = x^2$ at a point x.

Solution. We have

$$\lim_{t \to x} \frac{t^2 - x^2}{t - x} = \lim_{t \to x} x + t = 2x. \qquad \square$$

Exercise 2

Using the definition, calculate the derivative of $f(x) = x^3$ at a point x.

Solution. Using the formula $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ we have

$$\lim_{t \to x} \frac{t^3 - x^3}{t - x} = \lim_{t \to x} x^2 + xt + t^2 = 3x^2. \qquad \square$$

Example

Exercise 3

Using the definition, calculate the derivative of $f(x) = \sqrt{x}$ at a point x.

Solution. Using the formula

$$x - y = (\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}),$$

we obtain

$$\lim_{t \to x} \frac{\sqrt{t} - \sqrt{x}}{t - x} = \lim_{t \to x} \frac{1}{\sqrt{t} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \qquad \square$$

Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$, then f is continuous at x.

Proof. Note that

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x}(t - x) \underset{t \to x}{\longrightarrow} f'(x) \cdot 0 = 0. \qquad \square$$

Remark 1

The converse of this theorem is **not** true.

- Let $f(x) = |x|$ but it is not differentiable at $x = 0$.

Remark 2

It is also possible to construct a continuous function on $\mathbb{R}$ which is not differentiable at any point of $\mathbb{R}$.

Arithmetic theorem for derivatives

Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be differentiable at $x \in [a, b]$. Then $f + g$, $f \cdot g$, and $\frac{f}{g}$ are differentiable at x and we have

- (a) $(f+g)'(x) = f'(x) + g'(x)$,
- (b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$,
- (c) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$, whenever $g(x) \neq 0$.

Proof of (a). It is clear, since

$$(f+g)'(x) = \lim_{t \to x} \frac{f(t) + g(t) - f(x) - g(x)}{t - x} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} + \lim_{t \to x} \frac{g(t) - g(x)}{t - x} = f'(x) + g'(x). \qquad \square$$

Proof of (b) and (c)

Proof of (b). Let $h = f \cdot g$, then

$$h(t) - h(x) = f(t)(g(t) - f(x)) + f(x)(f(t) - g(x)).$$

Thus

$$\begin{aligned}(f \cdot g)'(x) = h'(x) &= \lim_{t \to \infty} \frac{h(t) - h(x)}{t - x} \\ &= \lim_{t \to x} f(t) \frac{g(t) - g(x)}{t - x} + \lim_{t \to x} g(x) \frac{f(t) - f(x)}{t - x} \\ &= f(x)g'(x) + g(x)f'(x). \qquad \square\end{aligned}$$

Proof of (c). Let $h = \frac{f}{g}$ and observe

$$\frac{h(t) - h(x)}{t - x} = \frac{1}{g(x)g(t)} \left(g(x) \frac{f(t) - f(x)}{t - x} - f(x) \frac{g(t) - g(x)}{t - x} \right).$$

Letting $t \to x$ we obtain the desired claim. $\square$

Examples

Example 1

$f(x) = c \in \mathbb{R}$ for all $x \in \mathbb{R}$, then $f'(x) = 0$ for all $x \in \mathbb{R}$.

Example 2

$f(x) = x^n$, then $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$. Indeed,

$$x^n - y^n = (x - y)\left(x^{n-1} + x^{n-2}y + \ldots + xy^{n-2} + y^{n-1}\right),$$

thus

$$\frac{f(t) - f(x)}{t - x} = t^{n-1} + t^{n-2}x + \ldots + x^{n-2}t + x^{n-1} \to_{t \to x} nx^{n-1}.$$

Example 3

$f(x) = \frac{1}{x^n}$, $x \neq 0$, then $f'(x) = -\frac{nx^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}}$.

Examples

Example 4

Every polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ is differentiable.

Example 5

Every $R(x) = \frac{P(x)}{Q(x)}$, where P, Q are polynomials, is differentiable for all $x \in \mathbb{R}$ such that $Q(x) \neq 0$.

Exercise

Calculate $f'(x)$, where $f(x) = \sqrt{x} + 3x^4 + 5$.

Solution. Using the previous theorem and the fact that

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}, \quad (x^4)' = 4x^3, \quad (5)' = 0,$$

we obtain $f'(x) = \frac{1}{2\sqrt{x}} + 12x^3$. □

Leibniz theorem

Theorem (Chain rule)

Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)), \quad a \leq t \leq b,$$

then h is differentiable at x and

$$h'(x) = g'(f(x))f'(x).$$

The latter identity is called **the chain rule**.

Proof: 1/2

Let $y = f(x)$. By the definition of the derivative we have

$$f(t) - f(x) = (t - x)(f'(x) + u(t)),$$

$$g(s) - g(y) = (s - y)(g'(y) + v(s)),$$

where $t \in [a, b]$, $s \in I$, and

$$\lim_{t \to x} u(t) = 0 \qquad \text{and} \qquad \lim_{s \to y} v(s) = 0.$$

Let $s = f(t)$ and note that

$$\begin{aligned} h(t) - h(x) = g(f(t)) - g(f(x)) &= (f(t) - f(x))(g'(y) + v(s)) \\ &= (t - x)(f'(x) + u(t))(g'(y) + v(s)). \end{aligned}$$

Proof: 2/2

If $t \neq x$, then

$$\frac{h(t) - h(x)}{t - x} = (g'(y) + v(s))(f'(x) + u(t)).$$

Letting $t \to x$ we see

$$s = f(t) \underset{t \to x}{\longrightarrow} f(x) = y$$

by the continuity of f. Thus

$$\lim_{t \to x} \frac{h(t) - h(x)}{t - x} = g'(y)f'(x) = g'(f(x))f'(x). \qquad \square$$

Example

Exercise

Calculate $h'(x)$, where

$$h(x) = (x^5 + x^3)^{100}.$$

Solution. By the chain rule

$$h = f \circ g, \quad f(x) = x^{100}, \quad g(x) = x^5 + x^3,$$

so

$$f'(x) = 100x^{99}, \quad \text{and} \quad g'(x) = 5x^4 + 3x^2,$$
$$h'(x) = 100(5x^4 + 3x^2)(x^5 + x^3)^{99}.$$

Remark

Newton's binomial formula could be also used to calculate $h'(x)$, but the solution seems to be longer.

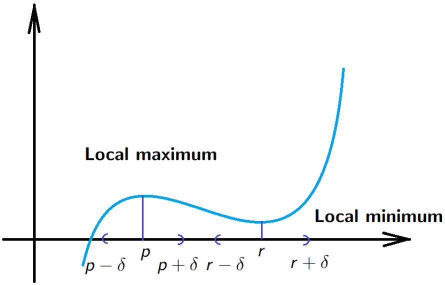
Local minimum and maximum

Local maximum and minimum

Let $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$. We say that f **has a local maximum at the point** $p \in X$ if there exists $\delta > 0$ such that

$$f(q) \leq f(p) \quad \text{for all} \quad q \in (p - \delta, p + \delta),$$

Local minimum is defined likewise.



Theorem

Theorem

If $f : [a, b] \to \mathbb{R}$ has a local maximum at $x \in (a, b)$ and if $f'(x)$ exists then $f'(x) = 0$. An analogous statement is also true for local minima.

Proof. If $x \in (a, b)$ is a local maximum then there exists $\delta > 0$ such that if $|q - x| < \delta$, then $f(q) \leq f(x)$.
We can assume that $a < x - \delta < x < x + \delta < b$ if $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \to x$ we see that $f'(x) \geq 0$. If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0.$$

Letting $t \to x$ then we obtain $f'(x) \leq 0$, thus we conclude $f'(x) = 0$. □

The mean-value theorem

The mean-value theorem

If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable in (a, b) then there is a point $x \in (a, b)$ at which

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x).$$

- Note that differentiability is not required at the endpoints.
- If $g(x) = x$, we recover the Lagrange theorem.

Lagrange theorem

$$\frac{f(b) - f(a)}{b - a} = f'(x) \quad \text{for some} \quad x \in (a, b).$$

Proof of the mean-value theorem: 1/2

- For $a \leq t \leq b$ consider

$$h(t) = (f(b) - f(a))g(t) - (g(b) - g(a))f(t).$$

- Then h is continuous on $[a, b]$ and h is differentiable in (a, b) and

$$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

- To prove the theorem we have to show that

$$h'(x) = 0 \quad \text{for some} \quad x \in (a, b).$$

- If h is constant, this holds for every $x \in (a, b)$.

Proof of the mean-value theorem: 2/2

Recall

A continuous function always attains its maximum and minimum on a compact set.

- If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point in $[a, b]$ for which h attains its maximum.
- Since $h(a) = h(b)$ then $x \in (a, b)$.
- By the previous theorem $h'(x) = 0$, since $h(x) = \sup_{y\in[a,b]} h(y)$.
- Similarly, if $h(t) < h(a)$ for some $t \in (a, b)$ the same argument applies, and we choose $x \in (a, b)$ where h attains its minimum.

This completes the proof of the theorem. □

Example

Exercise

Assume that f is differentiable, moreover

$$f(0) = 1, \qquad f(3) = 2.$$

Prove that there is $c \in [0, 3]$ such that $f'(c) = \frac{1}{3}$.

Solution. By the mean-value theorem

$$f(3) - f(0) = (3 - 0)f'(c)$$

for some $c \in (0, 3)$. Moreover, by our assumption,

$$1 = 2 - 1 = f(3) - f(0) = 3f'(c),$$

so $f'(c) = \frac{1}{3}$. □

Theorem

Theorem

Suppose f is differentiable in (a, b).

- If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.
- If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.
- If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

Proof. By the mean-value theorem for each $a < x_1 < x_2 < b$ we have

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1) \quad \text{for some} \quad x \in (x_1, x_2).$$

- If $f'(x) \geq 0$, then $f(x_2) \geq f(x_1)$.
- If $f'(x) = 0$, then $f(x_2) = f(x_1)$.
- If $f'(x) \leq 0$, then $f(x_2) \leq f(x_1)$.

□

Remark

Derivatives which exist at every point of an interval have an important property in common with functions which are continuous on the intervals:

their intermediate values are attained.

Theorem

Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable and suppose that

$$f'(a) < \lambda < f'(b).$$

Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

- A similar result holds of course if $f'(a) > f'(b)$.

Proof

Set $g(t) = f(t) - \lambda t$.

- Then $g'(a) < 0$ and $g(t_1) < g(a)$ for some $t_1 \in (a, b)$ since

$$0 > g'(a) = \lim_{a<t\to a} \frac{g(t) - g(a)}{\underbrace{t - a}_{>0}}.$$

- Similarly, since $g'(b) > 0$ we obtain $g(t_2) < g(b)$ for some $t_2 \in (a, b)$.
- Hence g attains its minimum on $[a, b]$ at some point $x \in (a, b)$.
- Hence we have $g'(x) = 0$, so $f'(x) = \lambda$ and we are done. □

Remark

If $f : [a, b] \to \mathbb{R}$ is differentiable then f' cannot have any simple discontinuity on $[a, b]$. But f' may have discontinuities of the second kind.

L'Hôpital's rule

L'Hôpital's rule

Suppose that $f, g : (a, b) \to \mathbb{R}$ are differentiable in (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose that

$$\frac{f'(x)}{g'(x)} \xrightarrow[x \to a]{} A. \qquad (*)$$

- (a) If $f(x) \xrightarrow[x \to a]{} 0$ and $g(x) \xrightarrow[x \to a]{} 0$, or
- (b) if $g(x) \xrightarrow[x \to a]{} +\infty$, then

$$\frac{f(x)}{g(x)} \xrightarrow[x \to a]{} A.$$

Remark

An analogous statement is true if $x \to b$ or if $g(x) \to -\infty$.

Proof: 1/4

Proof. We first consider the case $-\infty \leq A < +\infty$.

- Choose a real number q such that $A < q$ and the choose r such that $A < r < q$.
- By (*) there is $c \in (a, b)$ such that $a < x < c$ implies

$$\frac{f'(x)}{g'(x)} < r.$$

- If $a < x < y < c$ then the mean-value theorem shows that there is a point $t \in (x, y)$ such that

(**)

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

Proof: 2/4

- If $f(x) \underset{x \to a}{\longrightarrow} 0$ and $g(x) \underset{x \to a}{\longrightarrow} 0$ then we see

$$\frac{f(y)}{g(y)} \leq r < q, \quad \text{whenever} \quad a < y < c.$$

- If $g(x) \underset{x \to a}{\longrightarrow} +\infty$. Keeping y fixed in (**) we can choose a point $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Then

$$\frac{g(x) - g(y)}{g(x)} > 0.$$

Thus

$$\begin{aligned} \frac{f(x) - f(y)}{g(x)} &= \frac{f(x) - f(y)}{g(x) - g(y)} \frac{g(x) - g(y)}{g(x)} \\ &< r \frac{g(x) - g(y)}{g(x)} = r - \frac{g(y)}{g(x)} r. \end{aligned}$$

Proof: 3/4

- Hence

$$\frac{f(x)}{g(x)} < r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}, \quad \text{whenever} \quad a < x < c_1.$$

- If we let $x \to a$ (since $g(x) \xrightarrow[x \to a]{} +\infty$) we find $c_2 \in (a, c_1)$ such that

$$\frac{f(x)}{g(x)} < q, \quad \text{whenever} \quad a < x < c_2.$$

- We conclude that for any $q > A$ there is c_2 such that

$$a < x < c_2 \quad \text{implies} \quad \frac{f(x)}{g(x)} < q.$$

Proof: 4/4

- In the same manner if $-\infty < A \leq +\infty$ and p is chosen so that $p < A$ we can find a point c_3 such that

$$a < x < c_3 \quad \text{implies} \quad p < \frac{f(x)}{g(x)}.$$

- If $-\infty < A < +\infty$ we take $\varepsilon > 0$ and set $p = A - \varepsilon$, $q = A + \varepsilon$.
- Then there is c_3 so that for $a < x < c_3$ we have

$$A - \varepsilon < \frac{f(x)}{g(x)} < A + \varepsilon.$$

This completes the proof of the L'Hôpital rule. □

Derivatives of higher order

Second derivative

If f has a derivative f' on an interval and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' **the second derivative of f**.

- Continuing this way, we obtain:

$$f, f', f'', f^{(3)}, \dots, f^{(n)}, \dots$$

  each of which is derivative of the proceeding one.
- $f^{(n)}$ is called **the n-th derivative**, or **derivative of order n of f**.

Remark

- In order for $f^{(n)}(x)$ to exists at point x, $f^{(n-1)}(t)$ must exists in a neighbourhood of x (or in a one-sided neighborhood, if x is an endpoint of the interval on which f is defined) and $f^{(n-1)}$ must be differentiable at x.

Examples

Example

Consider $f(x) = x^n$ for $n \in \mathbb{N}$. Then

$$f'(x) = nx^{n-1},$$

$$f''(x) = n(n-1)x^{n-2},$$

$$f'''(x) = n(n-1)(n-2)x^{n-3},$$

$$\vdots$$

$$f^{(n)}(x) = n!.$$

Convex functions

Convex function

A function $f : (a, b) \to \mathbb{R}$ is **convex** if for every $x, y \in (a, b)$ one has

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

whenever $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$.

Observation 1

If $f : (a, b) \to \mathbb{R}$ is convex and if $a < s < t < u < b$, then

$$\frac{f(t) - f(s)}{t - x} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof. Since $s < t < u$ then we may write $t = \alpha u + \beta s$ for some $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$.

Proof

More precisely,

$$t = \alpha u + \beta s = \underbrace{\frac{t-s}{u-s}}_{=\alpha} u + \underbrace{\frac{u-t}{u-s}}_{=\beta} s.$$

Then by the **convexity**

$$f(t) = f\left(\frac{t-s}{u-s}u + \frac{u-t}{u-s}s\right) \le \frac{t-s}{u-s}f(u) + \frac{u-t}{u-s}f(s).$$

Hence

$$f(t) - f(s) \le \frac{t-s}{u-s}f(u) + \frac{u-t}{u-s}f(s) - \frac{u-s}{u-s}f(s),$$

so

$$f(t) - f(s) \le \frac{t-s}{u-s}f(u) - \frac{t-s}{u-s}f(s).$$

Hence

$$\frac{f(t)-f(s)}{t-s} \le \frac{f(u)-f(s)}{u-s}. \qquad \square$$

Observation 2

Observation 2

If $f : (a,b) \to \mathbb{R}$ is convex then for any $\lambda_1, \ldots, \lambda_n \in [0,1]$ satisfying

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1,$$

we have

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n).$$

Proof. For $n = 2$ if follows from definition of convexity. Suppose that the statement is true for $n \geq 2$ and we show it also holds for $n+1$. Let $\lambda_1, \ldots, \lambda_{n+1} \in [0,1]$ so that $\lambda_1 + \ldots + \lambda_{n+1} = 1$. Note that

(*)

$$\sum_{k=1}^{n} \frac{\lambda_k}{1-\lambda_{n+1}} = \frac{1}{1-\lambda_{n+1}} \sum_{k=1}^{n} \lambda_k = \frac{1-\lambda_{n+1}}{1-\lambda_{n+1}} = 1.$$

Proof

Then

$$
\begin{aligned}
& f(\lambda_1 x + \ldots + \lambda_{n+1} x_{n+1}) \\
&= f\left(\lambda_{n+1} x_{n+1} + (1-\lambda_{n+1})\left(\sum_{k=1}^{n} \frac{\lambda_k}{(1-\lambda_{n+1})} x_k\right)\right) \\
&\underbrace{\leq}_{\text{convexity}} \lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1}) f\left(\sum_{k=1}^{n} \frac{\lambda_k}{(1-\lambda_{n+1})} x_k\right) \\
&\underbrace{\leq}_{\text{induction+(*)}} \lambda_{n+1} f(x_{n+1}) + (1-\lambda_{n+1}) \sum_{k=1}^{n} \frac{\lambda_k}{(1-\lambda_{n+1})} f(x_k) \\
&= \sum_{k=1}^{n+1} \lambda_k f(x_k). \qquad \square
\end{aligned}
$$

Convexity and continuity

Theorem

If $f : (a, b) \to \mathbb{R}$ is convex then f is continuous on (a, b).

Proof. Let $a < s < u < v < t < b$. By Observation 1 one has

$$f(u) \leq f(s) + \frac{f(v) - f(s)}{v - s}(u - s)$$

and also

$$f(v) \leq f(u) + \frac{f(t) - f(u)}{t - u}(v - u).$$

Thus

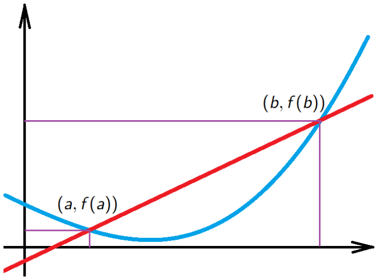
$$f(s) + \frac{f(u) - f(s)}{u - s}(v - s) \leq f(v) \leq f(u) + \frac{f(t) - f(u)}{t - u}(v - u).$$

Take $v = v_n$ for $n \in \mathbb{N}$. If $v_n \xrightarrow[n\to\infty]{} u$ converges to u we see that $\lim_{n\to\infty} f(v_n) = f(u)$ thus $\lim_{x\to u} f(x) = f(u)$. □

Sign of the second derivative

- The sign of the first derivative has been interpreted in terms of a geometric property of the function whether it is decreasing or increasing. We shall interpret the sign of the second derivative.
- Let $f : [a, b] \to \mathbb{R}$, then the equation of the line passing through $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$



Sign of the second derivative

- The condition that every point on the curve $y = f(x)$ lies below the line segment between $x = a$ and $x = b$ is that

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \quad \text{for} \quad a \leq x \leq b. \qquad (*)$$

- Any point x between a and b can be written in the form $x = a + t(b - a)$ with $t \in [0, 1]$. In fact, one sees that the map

$$t \to a + t(b - a)$$

  is a strictly increasing bijection between $[0, 1]$ and $[a, b]$.
- If we substitute the value of x in terms of t in (*) we find an equivalent inequality

$$f((1 - t)a + tb) \leq (1 - t)f(a) + tf(b),$$

  which is convexity of the function f on (a, b).

Second derivative test

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. Assume that f'' exists on (a, b) and $f''(x) > 0$ on (a, b). Then f is strictly convex on the interval $[a, b]$.

Proof. For $a < x < b$ we define

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) - f(x).$$

By the mean-value theorem we obtain

$$g'(x) = \frac{f(b) - f(a)}{b - a} - f'(x) = f'(c) - f'(x) \quad \text{for some} \quad a < c < b.$$

Using the mean-value theorem again for f' we find $g'(x) = f''(d)(c - x)$ for some d between c and x.

Proof

- If $a < x < c$, and using $f''(d) > 0$ we conclude that g is strictly increasing on $[a, c]$.
- If $c < x < b$ we conclude that g is strictly decreasing on $[c, b]$.
- Since $g(a) = 0$ and $g(b) = 0$ it follows $g(x) > 0$ when $a < x < b$, thus

$$f(x) < f(a) + \frac{f(b) - f(a)}{b - a}(x - a). \qquad \square$$

Concave function

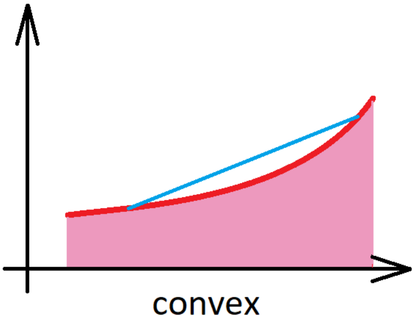
A function $f : (a, b) \to \mathbb{R}$ is concave if for every $x, y \in (a, b)$ one has

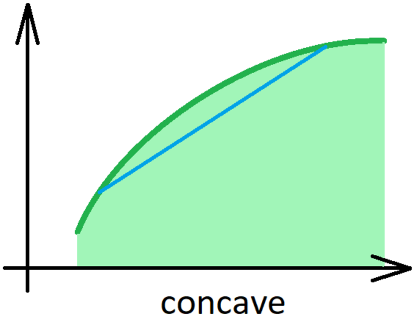
$$f(\alpha x + \beta y) \geq \alpha f(x) + \beta f(y)$$

whenever $\alpha, \beta \in [0, 1]$ and $\alpha + \beta = 1$.

- **Analogues of all above-proved theorems hold for concave functions in place of convex functions.**

Convex and concave functions





Theorem

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly increasing function. Then the inverse function of f is continuous and also strictly increasing.

Proof. Since f is continuous from the intermediate value theorem we know that the image of f is an interval, say $[\alpha, \beta] = f\big[[a, b]\big]$.

- Let $g : [\alpha, \beta] \to [a, b]$ be the inverse function. It is clear that g is also strictly increasing. We have to prove that g is continuous.
- Let $\gamma \in [\alpha, \beta]$. Given $\varepsilon > 0$ and $\gamma = f(x)$ consider the closed interval $[x_1, x_2]$, where

$$x_1 = \begin{cases} c - \varepsilon & \text{if } a \leq c - \varepsilon \\ a & \text{otherwise} \end{cases}, \qquad x_2 = \begin{cases} c + \varepsilon & \text{if } c + \varepsilon \leq b \\ b & \text{otherwise} \end{cases}.$$

  Then $f(x_1) \leq f(x_2)$.

Proof

- We assume $a < b$. We select

$$\delta = \min(f(x_2) - f(c), f(c) - f(x_1)).$$

- Suppose that $\delta > 0$. If $|y - \gamma| < \delta$, then there is unique x such that $y = f(x)$ and $x_1 < x < x_2$ and hence $|g(x) - c| < \varepsilon$.
- If $\delta = 0$, then either $a = c$ or $b = c$, that is c is an endpoint.
- Say $c = a$. In this case we disregard x_1 and let $\delta = f(x_2) - f(c)$.
- The same argument works if $c = b$ (we let $\delta = f(c) - f(x_1)$). □

Theorem

Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous and $a < b$. Assume that f is differentiable on (a, b) and $f'(x) > 0$ for $x \in (a, b)$. Then the inverse function g of f defined on $[\alpha, \beta] = f\big[[a, b]\big]$ is differentiable on (α, β) and

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))} \quad \text{for} \quad y \in (\alpha, \beta).$$

Proof. Let $\alpha < y_0 < \beta$ and $y_0 = f(x_0)$ and $y = f(x)$. Then

$$\frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} \xrightarrow[y \to y_0]{} \frac{1}{f'(x_0)} = \frac{1}{f'(g(y_0))}.$$

If $y \to y_0$ then $x \to x_0$ since g is continuous. □