Lesson 22

Exponential Function and Natural Logarithm Function, Power Series and Taylor's theorem

MATH 311, Section 4, FALL 2022

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The exponential function

The exponential function

We define

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \text{for} \quad z \in \mathbb{R}.$$

- Observe that $|E(z)| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} < \infty$. Thus the ratio test shows that the series converges absolutely for any $z \in \mathbb{R}$.

Recall

If $\sum_{n=0}^{\infty} a_n$ converges absolutely, $\sum_{n=0}^{\infty} a_n = A$, and $\sum_{n=0}^{\infty} b_n = B$, and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Then $\sum_{k=0}^{\infty} c_k = AB$.

Properties of the exponential function 1/4

Applying this result to absolutely convergent series $E(z)$, $E(w)$ we obtain

(*)

$$E(z)E(w) = E(z+w) \quad \text{for} \quad z, w \in \mathbb{R}.$$

Proof of (*). Indeed,

$$E(z)E(w) = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) \underbrace{=}_{Recall} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = E(z+w).$$

In the last line we have used the Binomial theorem. □

Properties of the exponential function 2/4

As the consequence we obtain

(**)

$$E(z)E(-z) = E(z-z) = E(0) = 1.$$

- This shows that $E(z) \neq 0$ for all $z \in \mathbb{R}$.
- We have $E(x) > 0$ if $x > 0$, giving $E(x) > 0$ for all $x \in \mathbb{R}$ by (**).
- It is easy to see that

$$\lim_{x\to\infty} E(x) = +\infty \quad \text{since} \quad E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- Consequently by (**) we obtain

$$\lim_{x\to\infty} E(-x) = 0 \quad \text{since} \quad E(-x) = \frac{1}{E(x)}.$$

Properties of the exponential function 3/4

- If $0 < x < y$ then

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} < \sum_{n=0}^{\infty} \frac{y^n}{n!} = E(y).$$

- Since $E(x)E(-x) = 1$ thus

$$E(-y) < E(-x),$$

  hence E is strictly increasing on $\mathbb{R}$.

- If $x \in \mathbb{R}$ then

$$E'(x) = \lim_{h\to 0} \frac{E(x+h) - E(x)}{h} = E(x) \underbrace{\lim_{h\to 0} \frac{E(h) - 1}{h}}_{=1} = E(x).$$

Properties of the exponential function 4/4

- Indeed,

$$\frac{E(h)-1}{h} = \frac{1}{h}\sum_{n=1}^{\infty}\frac{h^n}{n!} = \sum_{n=1}^{\infty}\frac{h^{n-1}}{n!},$$

hence

$$\begin{aligned}\left|\frac{1}{h}(E(h)-1)-1\right| &\leq \sum_{n=2}^{\infty}\frac{|h|^{n-1}}{n!} = |h|\sum_{n=2}^{\infty}\frac{|h|^{n-2}}{n!} \\ &\leq |h|E(|h|) \underbrace{\leq}_{|h|\leq 1} |h|e \xrightarrow[h\to 0]{} 0.\end{aligned}$$

- We have proved that $E'(x) = E(x)$ for all $x \in \mathbb{R}$.
- In particular, E is continuous on $\mathbb{R}$.

In the next theorem we summarize what we have proved.

Theorem

Theorem

We know that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = E(x).$$

The exponential function $\mathbb{R} \ni x \mapsto e^x$ satisfies the following properties:

(a) e^x is continuous and differentiable for all $x \in \mathbb{R}$,

(b) $(e^x)' = e^x$,

(c) e^x is strictly increasing on $\mathbb{R}$ and $e^x > 0$ for all $x \in \mathbb{R}$,

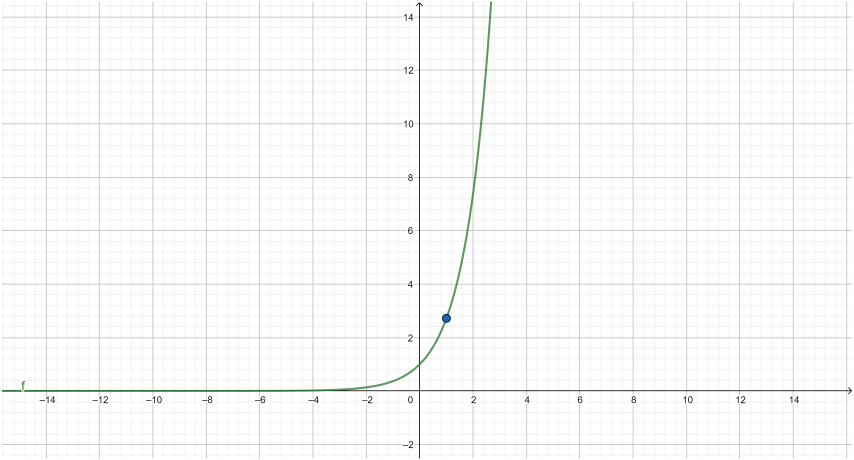
(d) $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$,

(e) $\lim_{x\to+\infty} e^x = +\infty$ and $\lim_{x\to-\infty} e^x = 0$,

(f) $\lim_{x\to+\infty} x^{-n} e^x = 0$ for all $n \in \mathbb{N}$.

Proof. We have proved (a)-(e). We only prove (f).

Graph of $f(x) = e^x$



Proof of (f)

Note that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} > \frac{x^{n+1}}{(n+1)!},$$

so that

$$x^n e^{-x} < \frac{(n+1)!}{x} \xrightarrow[x\to\infty]{} 0,$$

which gives the desired claim. □

Remark

Item (f) says that e^x tends to $+\infty$ faster that any polynomial.

- If $P(x) = \sum_{k=0}^{n} c_k x^k$, where $c_1, \ldots, c_n \in \mathbb{R}$, then

$$0 \leq \left| \frac{P(X)}{e^x} \right| \leq \frac{\sum_{k=0}^{n} |c_k| x^k}{e^x} \xrightarrow[x\to\infty]{} 0.$$

The logarithm function 1/4

- Since the exponential function $E(x) = e^x$ is strictly increasing and differentiable on $\mathbb{R}$ it has an inverse function L, which is also strictly increasing and differentiable and whose domain is $E[\mathbb{R}] = (0, \infty)$.
- L is defined by

$$E(L(y)) = y \quad \text{for all} \quad y > 0$$

  or, equivalently, $L(E(x)) = x$ for all $x \in \mathbb{R}$.
- Differentiating the latter equation

$$1 = (x)' = (L(E(x)))' = L'(E(x))E'(x) = L'(E(x))E(x).$$

  Thus $L'(E(x)) = \frac{1}{E(x)}$, hence

$$L'(y) = \frac{1}{y} \quad \text{for all} \quad y > 0.$$

The logarithm function 2/4

- Writing $u = E(x)$ and $v = E(y)$ note that

$$\begin{aligned} L(uv) = L(E(x)E(y)) &= L(E(x+y)) \\ &= x + y = L(u) + L(v) \quad \text{for} \quad u, v > 0. \end{aligned}$$

- From now on we will write $\log(x) = L(x)$.
- Since $\lim_{x\to+\infty} e^x = +\infty$ and $\lim_{x\to-\infty} e^x = 0$, we conclude

$$\lim_{x\to\infty} \log(x) = +\infty, \quad \text{and} \quad \lim_{x\to 0} \log(x) = \infty.$$

The logarithm function 3/4. Definition of x^α

- Since $x = E(L(x))$, it is easily seen that

$$x^n = E(nL(x)) \quad \text{and} \quad x^{1/m} = E\left(\frac{1}{m}L(x)\right) \quad \text{for} \quad n, m \in \mathbb{N}.$$

Thus

$$x^\alpha = E(\alpha L(x)) \quad \text{if} \quad \alpha \in \mathbb{Q}.$$

- It also makes sense to define

$$x^\alpha = E(\alpha L(x)) \quad \text{for} \quad \alpha \in \mathbb{R} \quad \text{and} \quad x > 0.$$

- The continuity and monotonicity of E and L show that everything makes sense and this definition coincides with

$$x^\alpha = \sup\{x^p \ : \ p < \alpha, \ p \in \mathbb{Q}\} \quad \text{if} \quad \alpha \in \mathbb{R} \quad \text{and} \quad x > 1.$$

The logarithm function 4/4

- If we differentiate

$$x^\alpha = E(\alpha a L(x)),$$

then

$$(x^\alpha)' = E'(\alpha L(x))\frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

- Finally note that

$$\lim_{x\to\infty} x^{-\alpha}\log(x) = 0 \quad \text{for every} \quad \alpha > 0.$$

That is, $\log(x)$ tends to $+\infty$ slower that any power of x.

- Indeed, since $x^\alpha \xrightarrow[x\to\infty]{} +\infty$, by L'Hôpital's rule

$$\lim_{x\to\infty}\frac{\log(x)}{x^\alpha} \underbrace{=}_{L'Hopital} \lim_{x\to\infty}\frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim_{x\to\infty}\frac{1}{\alpha x^\alpha} = 0.$$

Power series

Power series

Given a sequence $(c_n)_{n\in\mathbb{N}_0}$, where $c_n \in \mathbb{R}$, the series

$$\sum_{n=0}^{\infty} c_n x^n, \quad x \in \mathbb{R}$$

is called **a power series**.

- The numbers c_n are called **the coefficients of the series**.

Example 1

$\sum_{n=0}^{\infty} x^n$.

Example 2

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Radius of convergence

Radius of convergence

Given the power series

$$\sum_{n=0}^{\infty} c_n x^n$$

set

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad \text{and} \quad R = \frac{1}{\alpha}.$$

If $\alpha = 0$, then $R = +\infty$.

- The number R is called **the radius of convergence of** $\sum_{n=0}^{\infty} c_n x^n$.

Theorem

Theorem

The series $\sum_{n=0}^{\infty} c_n x^n$

- converges if $|x| < R$, and
- diverges if $|x| > R$.

Proof. Consider $a_n = c_n x^n$ and apply the root test

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|x|}{R}. \qquad \square$$

Example 1

$\sum_{n=0}^{\infty} n^n x^n$ has $R = 0$

Example 2

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = +\infty$.

Examples

Example 2

Hence if $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $R = +\infty$ since

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0.$$

Example 3

$\sum_{n=0}^{\infty} x^n$ has $R = 1$. If $|x| = 1$ the series $\sum_{n=0}^{\infty} x^n$ diverges. We also know

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad |x| < 1.$$

Examples

Example 4

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ has $R = 1$. If $x = 1$ the series diverges since

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

If $x = -1$ then the series converges since

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right| < \infty.$$

Taylor's theorem

Taylor's theorem

Suppose $f : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let $\alpha, \beta \in [a, b]$ be distinct and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

then there exists a point $x \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)(x)}}{n!}(\beta - \alpha)^n \qquad (*)$$

Proof 1/3

Remark

For $n = 1$ this is just **the mean–value theorem**. In general, the theorem says that f can be approximated by a polynomial of degree $n - 1$ and that (*) allows us to estimate the error term if we know bounds on $|f^{(n)}(x)|$.

Proof. Let M be a number such that

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n.$$

For $a \leq t \leq b$ set

$$g(t) = f(t) - P(t) - M(t - \alpha)^n.$$

- We have to show $n!M = f^{(n)}(x)$ for some $x \in (\alpha, \beta)$.

Proof 2/3

- Since
$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$
we have that $P^{(n)}(t) = 0$. Thus
$$g^{(n)}(t) = f^{(n)}(t) - n!M \quad \text{for} \quad t \in (\alpha, \beta)$$
(since $(x^n)^{(n)} = n!$).
- The proof will be completed if we show that $g^{(n)}(x) = 0$ for some $x \in (\alpha, \beta)$. Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, 1, 2, \ldots, n-1$, hence we have
$$g(\alpha) = g'(\alpha) = \ldots = g^{(n-1)}(\alpha) = 0.$$
Our choice of M shows that $g(\beta) = 0$.

Proof 3/3

- Hence by **the mean-value theorem**

$$g'(x_1) = 0 \quad \text{for some} \quad x_1 \in (\alpha, \beta)$$

since $0 = g(\alpha) - g(\beta) = (\beta - \alpha)g'(x_1)$.

- Using that $g'(\alpha) = 0$ we continue and obtain

$$0 = g'(x_1) - g'(\alpha) = (x_1 - \alpha)g''(x_2) \quad \text{for some} \quad \alpha < x_2 < x_1.$$

Thus $g''(x_2) = 0$.

- Repeating the previous arguments, after n steps we obtain

$$g^{(n)}(x_n) = 0 \quad \text{for some} \quad \alpha < x_n < x_{n-1} < \ldots < x_1 < \beta.$$

This completes the proof. □

Theorem

Theorem (Taylor's expansion formula)

Suppose that $f : [a, b] \to \mathbb{R}$ is n-times continuously differentiable on $[a, b]$ and $f^{(n+1)}$ exists in the open interval (a, b). For any $x, x_0 \in [a, b]$ and $p > 0$ there exists $\theta \in (0, 1)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r_n(x),$$

where $r_n(x)$ is **the Schlömlich–Roche remainder** function defined by

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!p}(1 - \theta)^{n+1-p}(x - x_0)^{n+1}.$$

Proof 1/3

- For $x, x_0 \in [a, b]$ set

$$r_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

- Wlog we may assume that $x > x_0$. For $z \in [x_0, x]$ define

$$\phi(z) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x - z)^k.$$

- We have $\phi(x_0) = r_n(x)$ and $\phi(x) = 0$, and ϕ' exists in (x_0, x) and

$$\phi'(z) = -\frac{f^{(n+1)}(z)}{n!}(x - z)^n.$$

Proof 2/3

- Indeed, by the telescoping we obtain

$$
\begin{aligned}
\phi'(z) &= -\left(\sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x-z)^k\right)' \\
&= -\sum_{k=0}^{n}\left(\frac{f^{(k+1)}(z)}{k!}(x-z)^k - \frac{f^{(k)}(z)}{k!}k(x-z)^{k-1}\right) \\
&= \sum_{k=1}^{n} \frac{f^{(k)}(z)}{(k-1)!}(x-z)^{k-1} - \sum_{k=0}^{n} \frac{f^{(k+1)}(z)}{k!}(x-z)^k \\
&= -\frac{f^{(n+1)}(z)}{n!}(x-z)^n.
\end{aligned}
$$

- Let $\psi(z) = (x-z)^p$, then ψ is continuous on $[x_0, x]$ with non-vanishing derivative on (x_0, x).

Proof 3/3

- By **the mean-value theorem**

$$\frac{\phi(x)-\phi(x_0)}{\psi(x)-\psi(x_0)} = \frac{\phi'(c)}{\psi'(c)} \quad \text{for some} \quad c \in (x_0, x).$$

- Thus, setting $c = x_0 + \theta(x - x_0)$,

$$r_n(x) = \underbrace{\phi(x_0)}_{=r_n(x)} - \underbrace{\phi(x)}_{=0} = -(\psi(x) - \psi(x_0))\frac{\phi'(c)}{\psi'(c)}$$

$$= \frac{f^{(n+1)}(c)}{n!}(x-c)^n \frac{-(x-x_0)^p}{-p(x-c)^{p-1}} =$$

$$= \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{pn!}(1-\theta)^{n+1-p}(x-x_0)^{n+1}. \quad \square$$

Corollary

Under the assumptions of the previous theorem.

Lagrange remainder

If $p = n + 1$ we obtain the Taylor formula with **the Lagrange remainder**:

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}.$$

Cauchy remainder

If $p = 1$ we obtain the Taylor formula with **the Cauchy remainder**:

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!}(1 - \theta)^n (x - x_0)^{n+1}.$$