Lesson 23

Applications of Calculus: Bernoulli's inequality and Weighted Mean Inequalities

MATH 311, Section 4, FALL 2022

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Theorem

Theorem (Taylor's expansion formula)

Suppose that $f : [a, b] \to \mathbb{R}$ is n-times continuously differentiable on $[a, b]$ and $f^{(n+1)}$ exists in the open interval (a, b). For any $x, x_0 \in [a, b]$ and $p > 0$ there exists $\theta \in (0, 1)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r_n(x),$$

where $r_n(x)$ is **the Schlömlich–Roche remainder** function defined by

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!p}(1 - \theta)^{n+1-p}(x - x_0)^{n+1}.$$

Corollary

Under the assumptions of the previous theorem.

Lagrange remainder

If $p = n + 1$ we obtain the Taylor formula with **the Lagrange remainder**:

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}.$$

Cauchy remainder

If $p = 1$ we obtain the Taylor formula with **the Cauchy remainder**:

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!}(1 - \theta)^n(x - x_0)^{n+1}.$$

Power series expansion for the logarithm

Theorem

For $|x| < 1$ we have

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k.$$

Proof. Note that $(\log(x+1))' = \frac{1}{x+1}$ and

$$(\log(x+1))'' = \left(\frac{1}{x+1}\right)' = -\frac{1}{(1+x)^2},$$

$$(\log(x+1))''' = \left(= -\frac{1}{(1+x)^2}\right) = \frac{2}{(1+x)^3},$$

$$(\log(x+1))^{(4)} = \left(\frac{2}{(1+x)^3}\right)' = -\frac{6}{(1+x)^4} = -\frac{3!}{(1+x)^4}.$$

Proof 1/2

Inductively, we have

$$(\log(1+x))^{(n)} = (-1)^{n+1}\frac{(n-1)!}{(x+1)^n}.$$

- We use the Taylor expansion formula at $x_0 = 0$ then

$$\log(1+x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + r_n(x) = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k}x^k + r_n(x),$$

since

$$f^{(0)}(0) = \log(1) = 0,$$
$$f^{(k)}(0) = (-1)^{k+1}(k-1)!.$$

Proof 2/2

- If $0 \le x < 1$ we use Lagrange's reminder. Then for some $0 < \theta < 1$,

$$|r_n(x)| = \left| \frac{f^{(n)}(\theta x)}{(n+1)!} x^{n+1} \right| = \frac{n!}{(n+1)!(1+\theta x)^n} x^{n+1} \le \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0.$$

- If $-1 < x < 0$ we use Cauchy's remainder. Then for some $0 < \theta < 1$,

$$|r_n(x)| = \left| \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!} (1-\theta)^n (x - x_0)^{n+1} \right|$$
$$= \left| \frac{n!}{n!(1+\theta x)^{n+1}} (1-\theta)^n x^{n+1} \right|.$$

- Since $-1 < \theta x < 0$, then $-\theta < \theta x$, so $1 - \theta < 1 + \theta x$, hence

$$|r_n(x)| \le \frac{(1-\theta)^n}{(1+\theta x)^{n+1}} |x|^{n+1} \le \frac{(1-\theta)^n}{(1-\theta)^{n+1}} |x|^{n+1} = \frac{|x|^{n+1}}{1-\theta} \xrightarrow[n\to\infty]{} 0$$

since $|x|^n \xrightarrow[n\to\infty]{} 0$ when $|x| < 1$. □

Newton's binomial formula

Theorem

If $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and $|x| < 1$ then

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \underbrace{\frac{\alpha(\alpha-1)\cdot \ldots \cdot(\alpha-n+1)}{n!}}_{\binom{\alpha}{n}} x^n.$$

This is called **Newton's binomial formula**.

Recall

For $n \in \mathbb{N}$ we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdot \ldots \cdot (n-k+1)}{k!}.$$

Proof 1/4

Proof. Let let $f(x) = (1+x)^\alpha$ and note that

$$f^{(n)}(x) = \alpha(\alpha-1)\cdot \ldots \cdot(\alpha-n+1)x^{\alpha-n}.$$

- Suppose first that $0 < x < 1$.
  Using the Lagrange remainder formula we have

$$r_n(x) = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}(1+x\theta)^{\alpha-n+1}.$$

Claim

For $|x| < 1$ we have

$$\lim_{n\to\infty} \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1} = 0.$$

Proof 2/4. Proof of the Claim.

- To prove the claim it suffices to use the following fact:

Fact

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = q < 1 \quad\Longrightarrow\quad \lim_{n\to\infty} a_n = 0$$

with $a_n = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n-1)x^{n+2}}{(n+2)!}\frac{(n+1)!}{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)x^{n+1}}\right|$$
$$= \left|\frac{\alpha-n-1}{n+2}x\right| \xrightarrow[n\to\infty]{} |x| < 1.$$

- Thus $r_n(x) \xrightarrow[n\to\infty]{} 0$ if we show that $(1+\theta x)^{\alpha-n-1}$ is bounded.

Proof 3/4

- Indeed, assuming that $0 < x < 1$ we see

$$(1+\theta x)^{-n} \leq 1,$$

- For $\alpha \geq 0$ we have

$$1 \leq (1+\theta x)^{\alpha} \leq (1+x)^{\alpha} \leq 2^{\alpha},$$

- For $\alpha < 0$ we have

$$2^{\alpha} \leq (1+x)^{\alpha} \leq (1+x\theta)^{\alpha} \leq 1$$

- Gathering all together we conclude that $(1+\theta x)^{\alpha-n-1}$ as desired.

Proof 4/4

- Now we assume that $-1 < x < 0$. Using the Cauchy remainder formula we have
$$r_n(x) = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}(1-\theta)^n(1+\theta x)^{\alpha-n-1}.$$
As before we show that $(1-\theta)(1+\theta x)^{\alpha-n-1}$ is bounded.
- Since $-1 < x < 0$ then $1+\theta x > 1-\theta$ and consequently
$$(1-\theta)^n \le (1-\theta)^n(1+\theta x)^{-n} = \frac{(1-\theta)^n}{(1+\theta x)^n} < 1.$$
- For $\alpha \le 1$ we have
$$1 \le (1+x\theta)^{\alpha-1} \le (1+x)^{\alpha-1}.$$
- For $\alpha \ge 1$ we have
$$(1+x)^{\alpha-1} \le (1+\theta x)^{\alpha-1} \le 1$$
and we are done. □

A function which does not have power series representation

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- It is not difficult to see that f is infinitely many times differentiable for any $x \in \mathbb{R}$.
- Moreover,

$$f^{(n)}(0) = 0 \quad \text{for any} \quad n \geq 0$$

  and $f(x) \neq 0$.
- Thus we see

$$0 \neq f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0.$$

Bernoulli's inequality: general form

Bernoulli's inequality: general form

For $x > -1$ and $x \neq 0$ we have

(a) $(1+x)^\alpha > 1 + \alpha x$ if $\alpha > 1$ or $\alpha < 0$,

(b) $(1+x)^\alpha < 1 + \alpha x$ if $0 < \alpha < 1$.

Proof. Applying Taylor's formula with the Lagrange remainder for $f(x) = (1+x)^\alpha$ we obtain

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} x^2.$$

Proof

- For $\alpha > 1$ or $\alpha < 0$ we have
$$\frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} > 0.$$
- For $0 < \alpha < 1$ we have
$$\frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} < 0.$$
- Consequently, for $\alpha > 1$ or $\alpha < 0$ we obtain
$$1 + \alpha x + \frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2}x^2 > 1 + x\alpha.$$
- Similarly, for $0 < \alpha < 1$, we obtain
$$1 + \alpha x + \frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2}x^2 > 1 + x\alpha.$$

This completes the proof. □

Proposition

Proposition

For $x > 0$ one has

$$\frac{x}{x+1} < \frac{2x}{x+2} \leq \log(x+1) < x.$$

Proof. Let $f(x) = x - \log(1+x)$, then

$$f(0) = 0,$$

$$f'(0) = 1 - \frac{1}{x+1} > 0 \quad \iff \quad x > 0$$

thus f is increasing for $x > 0$. Hence $f(x) > f(0)$ for $x > 0$, so

$$\log(1+x) < x.$$

Proof

We now consider

$$h(x) = \log(1+x) - \frac{2x}{x+1} \quad \text{for} \quad x > 0.$$

Note that $h(0) = 0$ and

$$h'(x) = \frac{x^2}{(x+1)(x+2)^2} > 0 \quad \text{for} \quad x > 0.$$

Thus h is increasing for $x > 0$ and

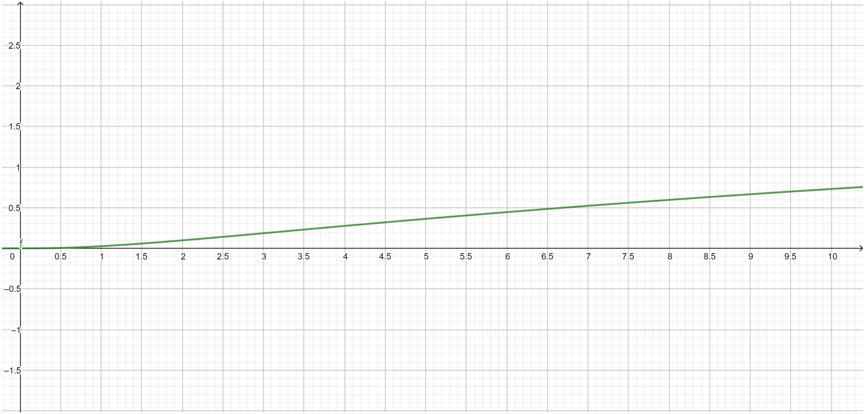
$$h(x) > h(0) = 0.$$

Consequently

$$\log(1+x) > \frac{2x}{x+2} > \frac{x}{x+1}$$

for $x > 0$ as desired. □

Graph of the function $\log(x+1) - \frac{2x}{x+2}$



Application

Application

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}\right)=\log 2.$$

Proof. Note that

$$\frac{1}{n+1}<\log\left(1+\frac{1}{n}\right)<\frac{1}{n}\quad \text{for}\quad n>1$$

upon taking $x=\frac{1}{n}$ in $\frac{x}{x+1}<\log(1+x)<x$. Consequently

$$\log\left(\frac{2n+1}{n}\right)<\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}<\log\left(\frac{2n}{n-1}\right).$$

Thus

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}\right)=\log 2. \qquad \square$$

Inequalities between weighted means

Theorem

If $x_1, \ldots, x_k > 0$ and $\alpha_1, \ldots, \alpha_k > 0$ and $\sum_{j=1}^{k} \alpha_j = 1$, then

$$x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k} \leq \alpha_1 x_1 + \ldots + \alpha_k x_k.$$

Proof. Let $f(x) = \log(x)$ and note that

$$f'(x) = \frac{1}{x} \quad \text{and} \quad f''(x) = \frac{-1}{x^2} < 0.$$

Thus $f''(x) < 0$ for all $x > 0$ which means that f is concave. In other words, for all $x_1, \ldots, x_k > 0$ and $\alpha_1, \ldots, \alpha_k > 0$ obeying condition $\alpha_1 + \ldots + \alpha_k = 1$, we have

$$f(\alpha_1 x_1 + \ldots + \alpha_k x_k) \geq \alpha_1 f(x_1) + \ldots + \alpha_k f(x_k).$$

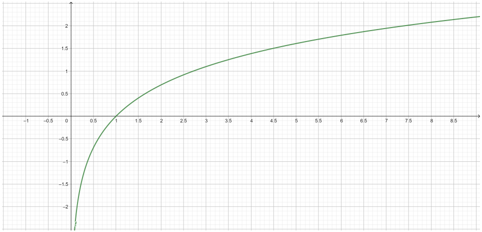
Proof

Consequently, we have

$$\log(x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k}) = \sum_{j=1}^{k} \alpha_j \log(x_j) \leq \log\left(\sum_{j=1}^{k} \alpha_j x_j\right)$$

if and only if

$$x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k} \leq \sum_{j=1}^{k} \alpha_j x_j. \qquad \square$$



Corollary

Corollary

If $p, q > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y > 0$, then

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

Proof. If suffices to apply the previous result with $\alpha_1 = \frac{1}{p}$, $\alpha_2 = \frac{1}{q}$ and $x_1 = x^p$, $x_2 = y^q$, then we obtain

$$xy = x_1^{1/p} x_2^{1/q} \leq \frac{1}{p}x_1 + \frac{1}{q}x_2 = \frac{1}{p}x^p + \frac{1}{q}y^q. \qquad \square$$

Remark

The inequality above is the key in the proof of Hölder's inequality.

Fibonacci sequence

Fibonacci sequence

The Fibonacci sequence $(f_n)_{n\in\mathbb{N}}$ is defined by

$$f_0 = 0, \quad f_1 = 1,$$

$$f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2.$$

Example

$$f_2 = 0 + 1 = 1,$$

$$f_3 = 1 + 1 = 2,$$

$$f_4 = 1 + 2 = 3,$$

$$f_5 = 2 + 3 = 5,$$

$$f_6 = 8, \quad f_7 = 13, \quad f_8 = 21.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 1/4

- Consider

$$\sum_{n=0}^{\infty} f_n x^n = x + \sum_{n=2}^{\infty}(f_{n-1}+f_{n-2})x^n$$
$$= x + x\sum_{n=2}^{\infty} f_{n-1}x^{n-1} + x^2\sum_{n=2}^{\infty} f_{n-2}x^{n-2}$$
$$= (x+x^2)\sum_{n=0}^{n=0} f_n x^n + x.$$

- Denoting $F(x) = \sum_{n=0}^{\infty} f_n x^n$ we have

$$F(x) = x + F(x)(x+x^2),$$

so

$$F(x) = \frac{x}{1-x-x^2}.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 2/4

- Then
$$1 - x - x^2 = -(x+\phi)(x+\psi),$$
where
$$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}.$$
- Then
$$F(x) = -\frac{x}{(x+\phi)(x+\psi)} = \frac{A}{x+\phi} + \frac{B}{x+\psi},$$
which is equivalent to
$$-x = A(x+\psi) + B(x+\phi).$$
- Hence
$$A = \frac{-\phi}{\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}}, \quad B = \frac{\psi}{\sqrt{5}} = \frac{1-\sqrt{5}}{2\sqrt{5}}.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 3/4

- So
$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{\psi}{x+\psi} - \frac{\phi}{x+\phi}\right).$$
- Recall that for $|x| < 1$ we have
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$
- Therefore
$$\frac{\psi}{x+\psi} = \frac{1}{1+\frac{x}{\psi}} = \frac{1}{1-x\phi} = \sum_{n=0}^{\infty} \phi^n x^n,$$
$$\frac{\phi}{x+\phi} = \sum_{n=0}^{\infty} \psi^n x^n.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 4/4

- Finally, we have

$$\sum_{n=0}^{\infty} f_n x^n = F(x)$$

$$= \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{\psi}{x+\psi} - \frac{\phi}{x+\phi}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty} \phi^n x^n - \sum_{n=0}^{\infty} \psi^n x^2\right) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}(\phi^n - \psi^n)x^n.$$

- Thus the formula for $(f_n)_{n\in\mathbb{N}}$ is given by

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$