Lesson 24

Power series of trigonometric functions done right

MATH 311, Section 4, FALL 2022

December 6, 2022

Discussion

Suppose that two functions $f, g : \mathbb{R} \to \mathbb{R}$ obeying the following condition

$$(*) \qquad f' = g, \qquad g' = -f, \qquad f(0) = 0, \qquad g(0) = 1 \qquad \text{exist.}$$

- We will show that they are **determined uniquely**. Note that

$$f^2(x) + g^2(x) = 1.$$

- Indeed, differentiating $f^2 + g^2$ we obtain

$$(f^2 + g^2)'(x) = 2(f'f + g'g)(x) = 2(fg - fg) = 0.$$

- Hence $f^2(x) + g^2(x) = C$, but $f(0) = 0$ and $g(0) = 1$, so $C = 1$.

Proof of the uniqueness 1/2

- Suppose that there are two functions $f_1, g_1 : \mathbb{R} \to \mathbb{R}$ obeying

$$f_1' = g_1, \qquad g_1' = -f_1, \qquad f_1(0) = 0, \qquad g_1(0) = 1.$$

- Our aim is to show that $f = f_1$ and $g = g_1$.
- Note that

$$(fg_1 - f_1 g)' = f' g_1 + f g_1' - f_1' g - f_1 g' = g g_1 - f f_1 - g g_1 + f_1 f = 0,$$

and

$$(f f_1 + g g_1)' = f' f_1 + f f_1' + g' g_1 + g g_1' = g f_1 + f g_1 - f g_1 - g f_1 = 0.$$

- Hence $fg_1 - f_1 g$ and $ff_1 + gg_1$ are constant functions and we have

$$\begin{cases} fg_1 - f_1 g = a & | \quad \cdot f \\ ff_1 + g_1 g = b & | \quad \cdot g, \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} f^2 g_1 - f f_1 g = af \\ f f_1 g + g_1 g^2 = bg. \end{cases}$$

Proof of the uniqueness 2/2

- Adding the equations and using $f^2 + g^2 \equiv 1$ we get $g_1 = af + bg$.
- Similarly

$$\begin{cases} fg_1 - f_1 g = a & | \quad \cdot g \\ ff_1 + g_1 g = b & | \quad \cdot f, \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} fgg_1 - f_1 g^2 = ag \\ f^2 f_1 + g_1 gf = bf. \end{cases}$$

- Subtracting the equations and using $f^2 + g^2 \equiv 1$ we get $f_1 = bf - ag$.
- Hence

$$\begin{cases} g_1 = af + bg \\ f_1 = bf - ag. \end{cases} \qquad (*)$$

Using $f(0) = f_1(0) = 0$ and $g(0) = g_1(0) = 1$ we get $a = 0$, $b = 1$ and, finally, we conclude that $f = f_1$, $g = g_1$. □

Heuristics 1/2

- If functions $f, g : \mathbb{R} \to \mathbb{R}$ obeying the following condition

$$(*) \qquad f' = g, \qquad g' = -f, \qquad f(0) = 0, \qquad g(0) = 1$$

exist, then they are differentiable infinitely many times.

- Since $f' = g$ and $g' = -f$, thus

$$f'' = -f, \quad g'' = -g, \quad f''' = -g, \quad g''' = f, \quad f^{(4)} = f, \quad g^{(4)} = g.$$

- Using Taylor's formula with Lagrange's remainder at $x_0 = 0$ one has

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + r_n(x), \qquad \text{where} \qquad r_n(x) = \frac{f^{(n+1)}(\theta x)}{(n+1)!} x^{n+1}.$$

$$g(x) = \sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} x^k + \widetilde{r}_n(x), \qquad \text{where} \qquad \widetilde{r}_n(x) = \frac{g^{(n+1)}(\theta x)}{(n+1)!} x^{n+1}.$$

Heuristics 2/2

- Since $f^2(x) + g^2(x) = 1$, so $|f(x)|, |g(x)| \leq 1$, and consequently

$$|f^{(n)}(x)| \leq 1 \qquad \text{and} \qquad |g^{(n)}(x)| \leq 1,$$

which implies

$$|r_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1} \qquad \text{and} \qquad |\widetilde{r}_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}.$$

- This in turn implies that

$$\lim_{n\to\infty} |r_n(x)| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} |\widetilde{r}_n(x)| = 0.$$

- Therefore if f, g exist then they are defined as the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad \text{and} \qquad g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Proof of the existence 1/3

- It make sense to define

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \text{and} \quad C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

- We show that

$$S'(x) = C(x), \quad C'(x) = -S(x), \quad S(0) = 0, \quad C(0) = 1.$$

- Obviously we have $S(0) = 0$ and $C(0) = 1$. We only have to show that $S'(x) = C(x)$ and $C'(x) = -S(x)$.
- Indeed, to prove $S'(x) = C(x)$, note that

$$\frac{1}{h}(S(x+h) - S(x)) = \frac{1}{h}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}[(x+h)^{2n+1} - x^{2n+1}]\right)$$

$$= \frac{1}{h}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left[\sum_{k=0}^{2n+1} \binom{2n+1}{k} h^k x^{2n+1-k} - x^{2n+1}\right]$$

Proof of the existence 2/3

- Thus

$$\begin{aligned}
&\frac{1}{h}(S(x+h)-S(x)) \\
&\quad = \frac{1}{h}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\sum_{k=1}^{2n+1}\binom{2n+1}{k}h^k x^{2n+1-k} \\
&\quad = \frac{1}{h}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[x^{2n}(2n+1)+\sum_{k=2}^{2n+1}\binom{2n+1}{k}h^k x^{2n+1-k}\right] \\
&\quad = \underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}}_{C(x)}+\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\sum_{k=2}^{2n+1}\binom{2n+1}{k}h^{k-1}x^{2n+1-k}}_{R(x)}.
\end{aligned}$$

Proof of the existence 3/3

- Recalling that

$$R(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \sum_{k=2}^{2n+1} \binom{2n+1}{k} h^{k-1} x^{2n+1-k}$$

- Then if $|h| \leq \frac{1}{2}$ one has

$$\begin{aligned} |R(x)| &\leq |h| \sum_{n=1}^{\infty} \frac{1}{(2n+1)!} \sum_{k=0}^{2n+1} \binom{2n+1}{k} |x|^{2n+1-k} \\ &\leq |h| \sum_{n=1}^{\infty} \frac{(|x|+1)^{2n+1}}{(2n+1)!} \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

- Hence $S'(x) = C(x)$. Similarly $C'(x) = -S(x)$.

Sine and Cosine functions

Theorem

There are unique functions $S, C : \mathbb{R} \to \mathbb{R}$ such that

$$S'(x) = C(x), \quad C'(x) = -S(x), \quad S(0) = 0, \quad C(0) = 1.$$

In other words, the functions S, C satisfy condition $(*)$, and they are given explicitly by the following formulas

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \text{and} \quad C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

They will be called respectively **sine** and **cosine** functions and will be denoted by $\sin(x) = S(x)$ and $\cos(x) = C(x)$.

Properties of $\sin(x)$ and $\cos(x)$

Properties

We have the following properties

(a) $\sin(x)^2 + \cos(x)^2 = 1$,

(b) $\sin(-x) = -\sin(x)$,

(c) $\cos(-x) = \cos(x)$,

(d) $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$,

(e) $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.

Proof of (a). It is clear since $\sin(x) = S(x)$ and $\cos(x) = C(x)$, and $S(0) = 0$, $C(0) = 1$ and

$$S'(x) = C(x) \quad \text{and} \quad C'(x) = -S(x).$$

Thus

$$S(x)^2 + C(x)^2 = 1.$$

Proof of (b) and (c)

Proof of (b).

$$\sin(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(-x)^{2n+1} = -\sin(x).$$

Proof of (c).

$$\cos(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(-x)^{2n} = \cos(x).$$

Proof of (d) and (e)

Proof of (d) and (e). Taking

$$f(x) = \sin(x), \qquad g(x) = \cos(x),$$

and

$$f_1(x) = \sin(x+y), \qquad g_1(x) = \cos(x+y),$$

for a fixed $y > 0$. Proceeding as in the proof of the uniqueness we obtain

$$\begin{cases} g_1 = af + bg \\ f_1 = bf - ag. \end{cases} \qquad (*)$$

Solving this equation one sees that

$$a = -\sin(y) \quad \text{and} \quad b = \cos(y).$$

This completes the proof of the theorem. □

Observations

Observation 1

- Since $(\sin(x))^2 + (\cos(x))^2 = 1$ thus

$$|\sin(x)| \leq 1,$$
$$|\cos(x)| \leq 1.$$

Observations

- The derivative of $\sin(x)$ at 0 is equal to 1, and the derivative is continuous. Thus it follows that the derivative of $\sin(x)$ (which is $\cos(x)$) is positive for all numbers in some open interval containing 0.
- Hence $\sin(x)$ is strictly increasing in such an interval and strictly positive for all $x > 0$ in such an interval.

Observations

Observation 3

- We shall prove that there is $x_0 > 0$ such that $\sin(x_0) = 1$, which means that $\cos(x_0) = 0$. Suppose that no such number exists.
- Since $\cos(x)$ is continuous, we conclude that $\cos(x)$ cannot be negative for any value of $x > 0$ by the intermediate value theorem.
- Hence $\sin(x)$ is strictly increasing for all $x > 0$ and $\cos(x)$ is strictly decreasing for all $x > 0$. Let $a > 0$. Then

$$0 < \cos(2a) = \cos(a)^2 - \sin(a)^2 < \cos^2(a).$$

- By induction

$$0 < \cos(2^n a) < (\cos(a))^{2^n} \quad \text{for all} \quad n \in \mathbb{N}.$$

- Hence $\lim_{n\to\infty} \cos(2^n a) = 0$ since $0 < \cos(a) < 1$.

Observations

Observation 3

- Since $\cos(x)$ is strictly decreasing for $x > 0$ it follows that $\cos(x)$ approaches to 0 as $x \to \infty$, and hence $\lim_{x\to\infty} \sin(x) = 1$.
- In particular, there is $b > 0$ so that

$$\cos(b) < \frac{1}{4} \quad \text{and} \quad \sin(b) > \frac{1}{2}.$$

- Then

$$0 < \cos(2b) = \cos^2(b) - \sin^2(b) < \frac{1}{16} - \frac{1}{4} < -\frac{3}{16} < 0,$$

which is a contradiction.
- **Thus there is $x_0 > 0$ so that $\cos(x_0) = 0$.** □

Observations

Observation 4

- By Observation 3

$$A = \{x > 0 \ : \ \cos(x) = 0\} = \{x > 0 \ : \ |\sin(x)| = 1\} \neq \emptyset.$$

- Let $c = \inf A$. By the continuity of $\cos(x)$ we see that

$$\cos(c) = 0 \quad \text{and} \quad |\sin(c)| = 1.$$

  and $c > 0$.
- We define $\pi = 2c$ thus $c = \frac{\pi}{2}$.
- Since $c = \inf A$ thus there is no $0 \leq x < \frac{\pi}{2}$ so that

$$\cos(x) = 0 \quad \text{and} \quad |\sin(x)| = 1.$$

Observations

Observation 4

- By **the intermediate value theorem** if follows that for $0 \leq x < \frac{\pi}{2}$ we have

$$0 \leq \sin(x) < 1 \quad \text{and} \quad 0 < \cos(x) \leq 1$$

and

$$\sin\frac{\pi}{2} = 1, \qquad \cos\frac{\pi}{2} = 0.$$

- Consequently

$$\sin(\pi) = 0 \quad \text{and} \quad \cos(\pi) = -1,$$

$$\sin(2\pi) = 0 \quad \text{and} \quad \cos(2\pi) = 1.$$

Observations

Observation 5

For all $x \in \mathbb{R}$ one has

(a) $\sin(x + \frac{\pi}{2}) = \cos(x)$,

(b) $\cos(x + \frac{\pi}{2}) = -\sin(x)$,

(c) $\sin(x + \pi) = -\sin(x)$,

(d) $\cos(x + \pi) = -\cos(x)$,

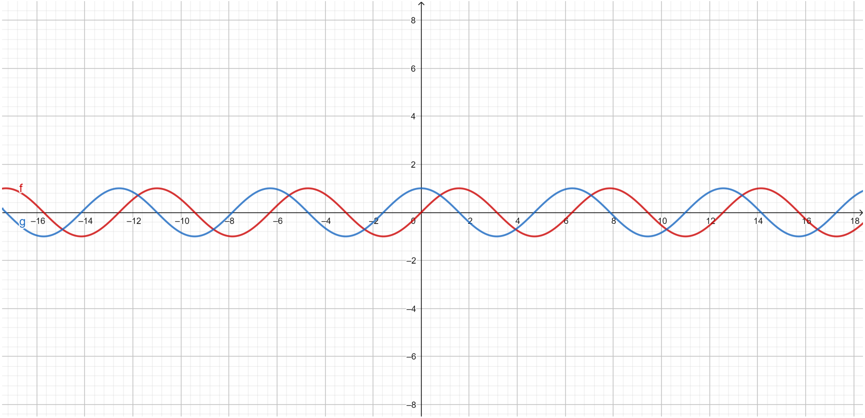
(e) $\sin(x + 2\pi) = \sin(x)$,

(f) $\cos(x + 2\pi) = \cos(x)$.

- The derivative of the $\sin(x)$ is positive for $0 < x < \frac{\pi}{2}$. Hence $\sin(x)$ is strictly increasing on $0 \leq x \leq \frac{\pi}{2}$.
- Similarly $\cos(x)$ is strictly decreasing on this interval. For $\frac{\pi}{2} \leq x \leq \pi$ we use the relation $\sin(x) = \cos(x - \frac{\pi}{2})$.
- For π to 2π we use $\sin(x) = -\sin(x - \pi)$.

Graphs of sin(x) and cos(x)

Now we can sketch the graphs of $\sin(x)$ and $\cos(x)$:



Periodic functions

Periodic function

A function $\phi : \mathbb{R} \to \mathbb{R}$ is **periodic** and a number $s > 0$ is called **a period of** ϕ if

$$\phi(x+s) = \phi(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

- We see that 2π is a period for $\sin(x)$ and $\cos(x)$.
- If s_1, s_2 are periods for ϕ then $s_1 + s_2$ is a period as well. Indeed,

$$\phi(x) = \phi(x+s_1) = \phi(x+s_1+s_2).$$

- If $s > 0$ is a period then $-s$ is a period as well

$$\phi(x) = \phi(x-s+s) = \phi(x-s).$$

Periods of $\sin(x)$ and $\cos(x)$

- Let s be a period for $\sin(x)$, then $\{m \in \mathbb{N} \,:\, 2\pi m \leq s\} \neq \emptyset$. Let

$$n = \max\{m \in \mathbb{N} \,:\, 2\pi m \leq s\}.$$

- Consider $t = s - 2\pi n$, then $0 \leq t < 2\pi$, t is also a period of $\sin(x)$. We must have

$$\sin(t + 0) = \sin(t) = 0, \quad \cos(0 + t) = \cos(0) = 1.$$

- From the known values of $\sin(x)$ and $\cos(x)$ for $0 \leq x \leq 2\pi$ it may only happen when $t = 0$.
- Thus $s = 2\pi n$.

Theorem

Theorem

Given a pair of numbers a, b such that $a^2 + b^2 = 1$, there exists a unique number $0 \leq t \leq 2\pi$ such that $a = \sin(t)$, $b = \cos(t)$.

Proof. We consider four different cases according to a, b are ≥ 0 or ≤ 0. In any case $|a| \leq 1$ and $|b| \leq 1$.

- Consider for instance the case

$$-1 \leq a \leq 0 \quad \text{and} \quad 0 \leq b \leq 1.$$

- By **the intermediate value theorem**, there is exactly one value of t such that $\frac{\pi}{2} \leq t \leq \pi$ and $\cos(t) = a$.
- We have $b^2 = 1 - a^2 = 1 - \cos^2(t) = \sin^2(t)$.
- Since $\frac{\pi}{2} \leq t \leq \pi$ then $\sin(t) \geq 0$ so b and $\sin(t) \geq 0$ and $b = \sin(t)$. Other cases follows similarly. □

Limit $\lim_{h \to 0} \frac{\sin(h)}{h}$

Theorem

We have
$$\lim_{h \to 0} \frac{\sin(h)}{h} = 0.$$

Proof. We have
$$\lim_{h \to 0} \frac{\sin(h)}{h} = \lim_{h \to 0} \frac{\sin(h) - \sin(0)}{h} = \sin'(0) = \cos(0) = 1. \qquad \square$$