Lesson 25

Riemann Integrals

MATH 311, Section 4, FALL 2022

December 9, 2022

Partitions

Partition

Let $[a, b]$ be a given interval. By a partition P of $[a, b]$ we mean a finite set of points

$$a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b.$$

Example 1

If $[a, b] = [0, 1]$, then $\{0, \frac{1}{2}, 1\}$ is a partition.

Example 2

If $[a, b] = [0, 1]$, then

$$\left\{ \frac{k}{n} \ : \ k = 0, 1, \ldots, n \right\}$$

is a partition for every $n \in \mathbb{N}$.

Upper and lower Riemann sums

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Corresponding to each partition P of $[a, b]$ we put

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad \text{and} \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x),$$

$$\Delta x_i = x_i - x_{i-1}.$$

Upper and lower Riemann sums

We define

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

- We always have that $L(P, f) \leq U(P, f)$.

Examples

Example 1

If $f(x) = x$ and $P = \{0, \frac{1}{2}, 1\}$, then

$$U(P,f) = \frac{1}{2}\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \frac{3}{4}, \quad \text{and} \quad L(f,L) = 0\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

Example 2

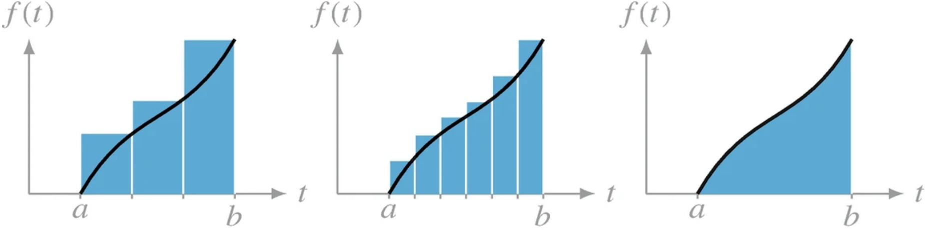
If $f(x) = x^2$ and

$$P = \left\{ \frac{k}{n} \ : \ k = 0, 1, \ldots, n \right\},$$

then

$$U(P,f) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2, \quad \text{and} \quad L(P,f) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i-1}{n}\right)^2.$$

Riemann sums - geometric interpretation



Upper and lower Riemann integral

Upper and lower Riemann integral

We define the **upper and lower Riemann integrals of f over** $[a, b]$ to be

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P, f), \quad \text{and} \quad \overline{\int_a^b} f(x)\,dx = \inf_P U(P, f),$$

where the inf and the sup are taken over all partitions P of $[a, b]$.

Riemann integral of f over $[a, b]$

If the upper and lower integrals are equal, we say that $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** on $[a, b]$, we write $f \in \mathcal{R}([a, b])$ and we denote the common value (which is called Riemann integral of f over $[a, b]$) by

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Riemann integral is well-defined

Fact

The upper and lower integrals are defined for every bounded function.

Proof. Let

$$m = \inf_{x \in [a,b]} f(x),$$
$$M = \sup_{x \in [a,b]} f(x).$$

Then

$$m \leq f(x) \leq M \quad \text{for all} \quad x \in [a, b].$$

Therefore

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

for every partition P. $\square$

Question of the integrability of f

Example

There is a bounded function f which is not integrable.

Proof. Let us define f on $[0, 1]$ to be

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Let us recall

Fact (*)

In any interval $[c, d]$ such that $c < d$ there is a rational and irrational number.

Proof

By the fact $(*)$, for every partition P of $[0,1]$, we have

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1,$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0.$$

Therefore

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P,f) = 0, \quad \text{and} \quad \overline{\int_a^b} f(x)\,dx = \inf_P U(P,f) = 1.$$

Hence $\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$ and f is not integrable. □

Refinements

Refinement

We say that the partition P^* is a refinement of P if $P^* \supseteq P$.

Common refinement

Given two partitions, P_1 and P_2,we say that P^* is their common refinement if

$$P^* = P_1 \cup P_2.$$

Example

If $[a, b] = [0, 2]$ and $P_1 = \{0, \frac{1}{2}, 1, 2\}$, $P_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{2}, 2\}$ are partitions, then their common refinement is

$$P^* = \left\{0, \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$

Theorem

Theorem

If P^* is a refinement of P, then

$$L(P,f) \le L(P^*,f),$$
$$U(P^*,f) \le U(P,f).$$

Proof. We prove the first statement.

- Suppose first that P^* contains just one point more than P. Let this extra point be x^*, and suppose

$$x_{i-1} \le x^* \le x_i \quad \text{for some} \quad i \in \{1,2,\ldots,n\}.$$

- Let

$$m_i = \inf_{x \in [x_{i-1},x_i]} f(x),$$
$$w_1 = \inf_{x \in [x_{i-1},x^*]} f(x), \quad \text{and} \quad w_2 = \inf_{x \in [x^*,x_i]} f(x)$$

Proof

- Then $w_1 \geq m_i$ and $w_2 \geq m_i$ and consequently

$$L(P^*, f) - L(P, f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1})$$
$$= (w_1 - m_i)(x^* - x_{i-1}) + (w_2 - m_i)(x_i - x^*) \geq 0.$$

- Finally, if P^* contains k points more than P, we repeat this reasoning k times. The proof of the second statement is analogous. □

Claim (*)

For two partitions P_1, P_2 of an interval $[a, b]$ one has

$$L(P_1, f) \leq U(P_2, f).$$

Proof. Let $P^* = P_1 \cup P_2$ be the common refinement of two partitions P_1 and P_2. By the previous theorem

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$

Theorem

Theorem

For any bounded function $f : [a, b] \to \mathbb{R}$ we have

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx.$$

Proof. By the Claim (*) for two partitions P_1, P_2 of an interval $[a, b]$ one has

$$L(P_1, f) \leq U(P_2, f).$$

Then

$$\underline{\int_a^b} f(x)\,dx = \sup_{P_1} L(P_1, f) \leq \inf_{P_2} U(P_2, f) = \overline{\int_a^b} f(x)\,dx,$$

This completes the proof of the theorem. □

Theorem

Theorem

A function $f \in \mathcal{R}([a, b])$ if and only if the following condition $(\mathcal{R})$ holds:

- For every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon. \qquad (\mathcal{R})$$

Proof. By the previous theorem, for every partition P we have

$$L(P, f) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq U(P, f).$$

Thus the condition $(\mathcal{R})$ implies

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(P, f) - L(P, f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, hence $f \in \mathcal{R}(\alpha)$.

Proof

Conversely, suppose that $f \in \mathcal{R}(\alpha)$. Then for every $\varepsilon > 0$ there are partitions P_1 and P_2 such that

$$\underline{\int_a^b} f(x)dx - L(P_1, f) < \frac{\varepsilon}{2} \quad \text{and} \quad U(P_2, f) - \overline{\int_a^b} f(x)dx < \frac{\varepsilon}{2}.$$

We choose P to be the **common refinement** of P_1 and P_2. Then

$$U(P, f) \leq U(P_2, f)$$
$$\leq \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2} = \int_a^b f(x)dx + \frac{\varepsilon}{2} = \underline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$$
$$\leq L(P_1, f) + \varepsilon \leq L(P, f) + \varepsilon.$$

This proves condition $(\mathcal{R})$ and completes the proof of the theorem. □

Theorem

Theorem (**)

If condition $(\mathcal{R})$ holds for $P = \{x_0, \ldots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i < \varepsilon.$$

Proof. Note that $f(s_i), f(t_i)$ lies in $[m_i, M_i]$, hence by the triangle inequality

$$|f(t_i) - f(s_i)| \leq \underbrace{M_i - m_i}_{\text{length}}.$$

Hence

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta x_i = \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon.$$

This completes the proof. □

Theorem

Theorem

If $f \in \mathcal{R}([a, b])$ and the hypotheses of $(**)$ hold, then

$$\left| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\, dx \right| < \varepsilon.$$

Proof. It is enough to note that

$$L(P, f) \leq \sum_{i=1}^{n} f(t_i)\Delta x_i \leq U(P, f),$$

$$L(P, f) \leq \int_a^b f(x)\, dx \leq U(P, f). \qquad \square$$

Riemann integrability for continuous functions

Theorem

If f is continuous on $[a, b]$ then $f \in \mathcal{R}([a, b])$.

Proof. Let $\varepsilon > 0$ be given. Choose $\eta > 0$ such that $\eta(b - a) < \varepsilon$. Recall that **if f is continuous on $[a, b]$, then it is also uniformly continuous.**

- Hence, there is $\delta > 0$ such that $|f(x) - f(t)| < \eta$ if $|t - x| < \delta$.
- In particular, that means that $M_i - m_i < \eta$ for every partition such that $\Delta x_i < \delta$.
- Hence,

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \eta \sum_{i=1}^{n} \Delta x_i = \eta(b - a) < \varepsilon.$$

The proof is completed. □

Riemann integrability for monotonic functions

Theorem

If $f : [a, b] \to \mathbb{R}$ is monotonic, then $f \in \mathcal{R}([a, b])$.

Proof. Let $\varepsilon > 0$ be given. For $n \in \mathbb{N}$ choose a partition P such that $\Delta x_i = \frac{b-a}{n}$. We suppose that f is monotonically increasing. Then

$$M_i - m_i = f(x_i) - f(x_{i-1}).$$

Hence, if n is taken large enough, we obtain

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = \frac{b-a}{n}(f(b) - f(a)) < \varepsilon, \end{aligned}$$

and we are done, the proof is analogous in the other case. □

Example

Example 1

Let

$$f(x) = \begin{cases} x & \text{for } x \in [0,1], \\ x^2 + 5 & \text{for } x \in (1,2], \\ x^3 + 9 & \text{for } x \in (3,4]. \end{cases}$$

Prove that f is Riemann integrable on $[0,4]$.

Solution. f is increasing, so f is Riemann integrable by the previous theorem.