Lesson 3

Least Upper Bounds and Greatest Lower Bounds, Axiom of Completeness, and Construction of $\mathbb{R}$ from $\mathbb{Q}$

MATH 311, Section 04, FALL 2022

September 13, 2022

Order on set

Order

Let S be a set. The **order** on S is the relation, denoted by $<$, with the following properties:

1. If $x, y \in S$, then **one and only one** of the following statements is true:
   - A. $x < y$,
   - B. $y < x$ (equivalently $x > y$),
   - C. $x = y$.
2. If $x, y, z \in S$, $x < y$ and $y < z$, then $x < z$.

Notation: $x \leq y$ means $(x = y$ or $x < y)$. Equivalently, $x \leq y$ is the negation of $x > y$.

Ordered set

Ordered set

Ordered set is the set S on which the order is defined.

Example

The set of rational numbers $\mathbb{Q}$ is ordered set if $<$ is the usual order on numbers. We say that $r < s$ for $r, s \in \mathbb{Q}$ iff $s - r > 0$.

Upper and lower bound

Upper bound

Suppose that S is an ordered set and $E \subseteq S$. If there is $\beta \in S$ such that $\alpha \leq \beta$ for all $\alpha \in E$, then E is **bounded above** and β is called the **upper bound** of E.

Lower bound

Suppose that S is an ordered set and $E \subseteq S$. If there is $\beta \in S$ such that $\beta \leq \alpha$ for all $\alpha \in E$, then E is **bounded below** and β is called the **lower bound** of E.

sup E

sup E

Suppose that S is an ordered set, $E \subset S$, and E is bounded from above. Suppose that there exists $\alpha \in S$ with the following properties

- **A** α is a upper bound of E,
- **B** if $\gamma < \alpha$, then γ is not an upper bound of E (equivalently, there is $x \in E$ such that $\gamma < x \leq \alpha$).

Then α is called **a least upper bound** or **supremum** of E. We write

$$\alpha = \sup E.$$

inf E

inf E

Suppose that S is an ordered set, $E \subset S$, and E is bounded from below. Suppose that there exists $\alpha \in S$ with the following properties

- A. α is a lower bound of E,
- B. if $\gamma > \alpha$, then γ is not a lower bound of E (equivalently, there is $x \in E$ such that $\alpha \leq x < \gamma$).

Then α is called **the greatest lower bound** or **infimum** of E. We write

$$\alpha = \inf E.$$

Examples

Example 1

Let

$$A = \{p \in \mathbb{Q} \ : \ p > 0, \ p^2 < 2\},$$
$$B = \{p \in \mathbb{Q} \ : \ p > 0, \ p^2 > 2\}.$$

The set A is bounded from above. In fact, the upper bounds of A are exactly the members of B. Since B contains no smallest member, it has no least upper bound in $\mathbb{Q}$.

Example 2

Let

$$E = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\}.$$

Then $\sup E = 1$ and $1 \in E$, $\inf E = 0$ and $0 \notin E$.

Least–upper–bound property

Least–upper–bound property

An ordered set S is said to have **least–upper–bound property** if the supremum $\sup E$ exists in S for all nonempty subsets $E \subseteq S$ that are bounded above.

Example 1

Let

$$A = \{p \in \mathbb{Q} \ : \ p > 0, \ p^2 < 2\}.$$

The previous slide shows that $\mathbb{Q}$ has not least–upper–bound property.

Theorem

Theorem

Suppose that S is an ordered set with the least–upper–bound property. Let $\emptyset \neq B \subseteq S$ be bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S and $\alpha = \inf B$.

Proof

Proof. Let

$$L = \{y \in S \colon y \leq x \text{ for all } x \in B\}.$$

We see that $L \neq \emptyset$, since B is bounded below. Every $x \in B$ is an upper bound of L. Thus L is bounded above and consequently the least–upper–bound property implies that $\alpha = \sup L$ exists in S.

We show that $\alpha \in L$. It suffices to prove that $\alpha \leq x$ for all $x \in B$. Suppose for a contradiction that there is $\gamma \in B$ such that $\gamma < \alpha$. By the definition of supremum γ is not an upper bound. Therefore, there exists $y \in L$ such that $\gamma < y \leq \alpha$, so $y \leq x$ for every $x \in B$, and hence $\gamma < x$ for all $x \in B$. In particular, we obtain $\gamma < \gamma$ since $\gamma \in B$, which is impossible!

Now we show that $\alpha = \inf B$. We have shown that $\alpha \in L$, which means that α is a lower bound of B, since $\alpha \leq x$ for all $x \in B$. If $\alpha < \beta$, then $\beta \notin L$ since α is an upper bound of L. If $\beta \notin L$ then there exists $x \in B$ such that $\beta > x \geq \alpha$. This proves that $\alpha = \inf B$. □

$\sup E$ and $\inf E$ - example

Example

Find $\sup E$ and $\inf E$, where

$$E = \{(-1)^n : n \in \mathbb{N}_0\}.$$

Solution. We have $(-1)^n = 1$ for even n and $(-1)^n = -1$ for odd n. Hence

$$E = \{-1, 1\}$$

$$\sup E = \max E = 1,$$
$$\inf E = \min E = -1.$$

□

sup E and inf E - example

Example

Find sup E and inf E, where

$$E = \left\{ \frac{1}{n^2+1} : n \in \mathbb{N}_0 \right\}.$$

Solution. Note that for all $n \in \mathbb{N}_0$ we have

$$\frac{1}{n^2+1} \leq \frac{1}{1}$$

and for $n = 0$ we have $\frac{1}{n^2+1} = \frac{1}{1}$. Hence $\sup E = 1$.
On the other hand, $\frac{1}{n^2+1} > 0$ for all $n \in \mathbb{N}_0$ and $\frac{1}{n^2+1}$ is small for large n (it will be formalized later). Hence $\inf E = 0 \notin E$. □

sup E and inf E - example

Example

Find sup E and inf E, where

$$E = \left\{ \frac{nm}{n^2 + m^2} : n, m \in \mathbb{N} \right\}.$$

Solution. Note that $\frac{nm}{n^2+m^2} > 0$ and for $m = 1$ we have

$$\frac{nm}{n^2 + m^2} = \frac{n}{1 + n^2} < \frac{1}{n}.$$

Hence $\inf E = 0$. On the other hand, for all $m, n \in \mathbb{N}$ we have

$$\frac{mn}{m^2 + n^2} \leq \frac{1}{2} \iff 2nm \leq m^2 + n^2 \iff 0 \leq (m - n)^2,$$

so $\frac{mn}{m^2+n^2} \leq \frac{1}{2}$ for all $m, n \in \mathbb{N}$. Moreover, $\frac{mn}{m^2+n^2} = \frac{1}{2}$ for $n = m = 1$. Hence $\sup E = \frac{1}{2}$. □

Field 1/2

Field

A **field** $\mathbb{F}$ is a set with two operations called **addition** $(+)$ and **multiplication** $((\,\cdot\,)$ or without symbol), which satisfies the following *field axioms* (A), (M), and (D).

Addition axioms (A)

- (A1) if $x, y \in \mathbb{F}$, then $x + y \in \mathbb{F}$,
- (A2) addition is commutative, i.e. $x + y = y + x$ for all $x, y \in \mathbb{F}$,
- (A3) addition is associative, i.e. $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$,
- (A4) $\mathbb{F}$ contains the element 0 such that $x + 0 = x$ for all $x \in \mathbb{F}$,
- (A5) to every $x \in \mathbb{F}$ corresponds an element $(-x) \in \mathbb{F}$ such that

$$x + (-x) = 0.$$

Field 2/2

Multiplication axioms (M)

- (M1) if $x, y \in \mathbb{F}$, then their product $xy \in \mathbb{F}$,
- (M2) multiplication is commutative, i.e. $xy = yx$ for all $x, y \in \mathbb{F}$,
- (M3) addition is associative, i.e. $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$,
- (M4) $\mathbb{F}$ contains the element $1 \neq 0$ such that $1x = x$ for all $x \in \mathbb{F}$,
- (M5) if $x \in \mathbb{F}$ and $x \neq 0$ then there exists an element $\frac{1}{x} \in \mathbb{F}$ such that

$$x \cdot \frac{1}{x} = 1.$$

Distributive law (D)

- (D1) $x(y + z) = xy + xz$ holds for all $x, y, z \in \mathbb{F}$.

Field properties - addition

Example 1

$\mathbb{Q}$ is a field.

Example 2

$\mathbb{Z}$ is not a field, because (M5) does not hold, i.e. there is no $x \in \mathbb{Z}$ such that $2x = 1$.

Properties of addition

The axioms of addition imply the following:

- (A) if $x + y = x + z$, then $y = z$,
- (B) if $x = x + y$, then $y = 0$,
- (C) if $x + y = 0$, then $y = (-x)$,
- (D) $(-(-x)) = x$.

Proofs

Proof of (A).

$$y \overbrace{=}^{(A4)} 0 + y \overbrace{=}^{(A5)} (-x + x) + y \overbrace{=}^{(A3)} -x + (x + y)$$
$$= -x + (x + z) \overbrace{=}^{(A3)} (-x + x) + z \overbrace{=}^{(A5)} 0 + z \overbrace{=}^{(A4)} z.$$

To prove (B), we take $z = 0$ in (A).

To prove (C) we take $z = -x$ in (A).

Since $x + (-x) = 0$, so by (C) with $-x$ in place of x we get

$$(-(-x)) = x.$$

□

Field properties - multiplication

Properties of multiplication

The axioms of multiplication imply the following:

- A) if $x \neq 0$ and $xy = xz$, then $y = z$,
- B) if $x \neq 0$ and $x = xy$, then $y = 1$,
- C) if $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$,
- D) if $x \neq 0$, then $\frac{1}{\frac{1}{x}} = x$

Exercise.

Further field properties

Properties of fields

The field axioms imply the following:

1. $x \cdot 0 = 0$ for all $x \in \mathbb{F}$,
2. if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$,
3. $(-x)y = -(xy) = x(-y)$ for all $x, y \in \mathbb{F}$,
4. $(-x)(-y) = xy$ for all $x, y \in \mathbb{F}$.

- For the proof of (A), we use (D1):

$$0x + 0x \overbrace{=}^{(D1)} (0 + 0)x = 0x.$$

Thus we must have $0x = 0$.

Proofs

- To prove (B) assume $x, y \neq 0$, but $xy = 0$. Then
$$1 = \frac{1}{x}\frac{1}{y}xy = \frac{1}{x}\frac{1}{y}0 = 0,$$
but $0 \neq 1$.
- To prove (C) we write
$$(-x)y + xy \overbrace{=}^{(D1)} (-x + x)y = 0y = 0,$$
thus $(-x)y = -(xy)$.
- To prove (D) we use (C) and we write
$$(-x)(-y) = -(x(-y)) = -(-(xy)) = xy.$$

□

Ordered field

Ordered field

An **ordered field** is a field with is also an ordered set such that

- A if $x, y, z \in \mathbb{F}$ and $y < z$, then $x + y < x + z$,
- B $xy > 0$ if $x > 0$ and $y > 0$.

Positive element

The element $x \in \mathbb{F}$ is called **positive** if $x > 0$.

Negative element

The element $x \in \mathbb{F}$ is called **negative** if $x < 0$.

Example

$\mathbb{Q}$ is an ordered field.

Properties of ordered fields

Proposition

The following are true in every ordered field:

A. if $x > 0$, then $-x < 0$ and vice versa,
B. if $x > 0$ and $y < z$, then $xy < xz$,
C. if $x < 0$ and $y < z$, then $xy > xz$,
D. if $x \neq 0$, then $x \cdot x = x^2 > 0$. In particular, $1 > 0$,
E. if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Proofs 1/2

Proof of (A). If $x > 0$, then $0 = -x + x > -x + 0$, thus $-x < 0$. If $x < 0$, then $0 = -x + x < -x + 0$, so that $-x > 0$.

Proof of (B). Since $z > y$ we have $z - y > y - y = 0$, hence $x(z - y) > 0$ if $x > 0$. Thus

$$xz = x(z - y) + xy > 0 + xy = xy.$$

Proof of (C). By (A),(B), and $(-x)y = -(xy) = x(-y)$:

$$-(x(z - y)) = (-x)(z - y) > 0$$

so that $x(z - y) < 0$ hence $xz < xy$.

Proofs 2/2

Proof of (D). If $x > 0$ we get $x^2 > 0$. If $x < 0$, then $-x > 0$, hence $(-x)^2 > 0$, but $x^2 = (-x)^2$. We also see $1^2 = 1$, thus $1 > 0$.

Proof of (E). If $y > 0$ and $v \leq 0$, then $yv \leq 0$. But $\frac{1}{y} \cdot y = 1 > 0$, thus $\frac{1}{y} > 0$. In similar way $\frac{1}{x} > 0$. Multiplying the inequality $x < y$ by $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$ we have

$$0 < \frac{1}{y} < \frac{1}{x}.$$

□

The real field

Subfield

We say A is **subfield** of B if A is a field and every element of A belongs to B.

Theorem

There exists an ordered field which has the least-upper-bound property and contains $\mathbb{Q}$ as a subfield. It will be called **the real field** and denoted by $\mathbb{R}$.

Axiom of completeness

We will say that the set of real numbers $\mathbb{R}$ satisfies axiom of completeness.

Axiom of completeness

Every non-empty set E of real numbers $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup E$ exists in $\mathbb{R}$.