# Lesson 4

## Consequences of the Axiom of Completeness, Decimals, Extended Real Number System

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# Archimedian property of $\mathbb{Q}$

## Archimedian property on $\mathbb{Q}$

1. Given any number $x \in \mathbb{Q}$ there exists $n \in \mathbb{N}$ satisfying

$$n > x.$$

2. Given any rational number $y > 0$ there exists an $n \in \mathbb{N}$ satisfying

$$\frac{1}{n} < y.$$

# Proof

The second property follows from the first one by taking $x = \frac{1}{y}$. Thus it suffices to prove the first statement. If $x \in \mathbb{Q}$ and $x \leq 0$, then there is nothing to do. Suppose that $x > 0$, then $x = \frac{p}{q}$ for some $p, q \in \mathbb{N}$. Consider the set

$$A = \{n \in \mathbb{N}_0 : n \leq x\}.$$

This set is nonempty since $x > 0$. We see that $m \in A$ iff $p - qm \geq 0$. Consider now the set

$$B = \{p - qn : n \in A\} \subset \mathbb{N}_0, \qquad \text{and} \qquad B \neq \emptyset.$$

By the well-ordering principle $B$ contains the smallest element, say $p - qm_0$ for some $m_0 \in A$. Thus for all $n \in A$ we have

$$p - qm_0 \leq p - qn \quad \Longleftrightarrow \quad n \leq m_0 \leq x.$$

Now we see that $x < m_0 + 1$ has desired property. □

# Example

## Example

Let $E = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\} \subset \mathbb{Q}$. Then

$$\sup E = 1 \quad \text{and} \quad 1 \in E,$$
$$\inf E = 0 \quad \text{and} \quad 0 \notin E.$$

- We now show that $\sup E = 1$. Of course $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$ and $1 \in E$, which shows that $\sup E = 1$.
- To prove that $\inf E = 0$ we note that $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$. Now take $x \in \mathbb{Q}$ such that $x > 0$. By the Archimedean property we always find $m \in \mathbb{N}$ such that $0 < \frac{1}{m} < x$ and we are done.

# Axiom of completeness AoC

Let us recall the axiom of completeness.

**Axiom of completeness AoC**

Every non-empty set of real numbers that is bounded above has a least upper bound.

Our goal is to apply the axiom of completeness to study some properties of real numbers.

# Application 1 - nested interval property

## Nested interval property

For each $n \in \mathbb{N}$, assume we are given a closed interval

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} \,:\, a_n \leq x \leq b_n\}.$$

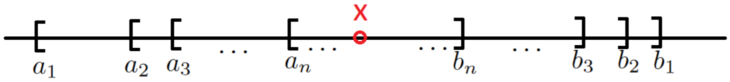
Assume also that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$. Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

has a nonempty intersection, that is

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset.$$

# Proof 1/2




Using (AoC) we will produce $x \in \mathbb{R}$ so that $x \in I_n$ for every $n \in \mathbb{N}$. Then

$$\bigcap_{n \in \mathbb{N}} I_n \supset \{x\} \neq \emptyset.$$

Consider the set $A = \{a_n \ : \ n \in \mathbb{N}\}$ of all left-hand endpoints of the intervals $I_n$. Because the intervals are nested one sees that every $b_n$ serves as an upper bound for $A$. Thus by the (AoC) we are allowed to write

$$x = \sup A \in \mathbb{R}.$$

The proof will be complete if we show $x \in I_n$ for all $n \in \mathbb{N}$.

# Proof 2/2

Since $x$ is an upper bound for $A$ thus

$$a_n \leq x \qquad \text{for all} \qquad n \in \mathbb{N}.$$

The fact that $b_n$ is an upper bound for $A$ and that $x$ is the least upper bound implies

$$x \leq b_n \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Thus

$$a_n \leq x \leq b_n$$

for all $n \in \mathbb{N}$ hence $x \in I_n$ for all $n \in \mathbb{N}$ and consequently

$$x \in \bigcap_{n \in \mathbb{N}} I_n.$$

□

# Application 2 - Archimedian property of $\mathbb{R}$

## Archimedian property

(i) Given any number $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying

$$n > x.$$

(ii) Given any real number $y > 0$ there exists an $n \in \mathbb{N}$ satisfying

$$\frac{1}{n} < y.$$

# Proof

**Proof.** Note that (ii) follows from (i) by letting $x = \frac{1}{y}$. It suffices to prove (i).
Without loss of generality we can assume that $x > 0$ and consider

$$A = \{nx \ : \ n \in \mathbb{N}\}.$$

Suppose for a contradiction that $A$ is bounded, i.e. there is $y \geq 0$ such that $nx \leq y$ for any $n \in \mathbb{N}$. This means that $y$ is an upper bound for $A$. By the (AoC):

$$\alpha = \sup A \in \mathbb{R}.$$

Since $x > 0$, $\alpha - x < \alpha$ and $\alpha - x$ is not upper bound of $A$. Thus we find $m \in \mathbb{N}$ such that

$$\alpha - x < mx \iff \alpha < (m+1)x.$$

This is contradiction since $\alpha$ is the supremum of $A$.

# Corollary - $\mathbb{Q}$ is dense in $\mathbb{R}$

## $\mathbb{Q}$ is dense in $\mathbb{R}$

If $x, y \in \mathbb{R}$ and $x < y$ then there is $p \in \mathbb{Q}$ such that $x < p < y$.

**Proof.** Since $x < y$, by Archimedian property there is $n \in \mathbb{N}$ such that

$$n(y - x) > 1.$$

- Then, we apply Archimedian property to find $m_1, m_2 \in \mathbb{Z}$ such that $m_1 > nx$ and $m_2 > -nx$. Then $-m_2 < nx < m_1$.
- Hence there is an integer $m$ with $-m_2 \leq m \leq m_1$ such that

$$m - 1 \leq nx < m.$$

- We combine these inequalities to get

$$nx < m \leq nx + 1 < ny, \qquad \text{so} \qquad x < p = \frac{m}{n} < y.$$

□

# More general

**Theorem 1.4.5**

For every real $x > 0$ and $n \in \mathbb{N}$ there is a unique real number $y > 0$ such that

$$y^n = x.$$

**Proof: Uniqueness.** The fact that there exists at most one such $y$ is clear, since $0 < y_1 < y_2$ implies

$$y_1^n < y_2^n.$$

# Identity $b^n - a^n$

In the proof (in the existence part), we will use the following identity

$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-2}b + a^{n-1})$$

which holds for all $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

# Proof: 1/3

- **Proof: Existence.** Let

$$E = \{t > 0 \ : \ t^n < x\}.$$

- If $t = \frac{x}{x+1}$, then $0 \leq t < 1$ hence

$$t^n \leq t < x$$

  thus $t \in E$ and $E \neq \emptyset$.
- If $t > x + 1$, then $t^n > t > x$, so that $t \notin E$. Thus $1 + x$ is an upper bound of $E$.
- By the (AoC) we may write $y = \sup E \in \mathbb{R}$. We will show that

$$y^n = x.$$

- It suffices to show that $y^n < x$ and $y^n > x$ cannot hold.

# Proof: 2/3. Case $y^n < x$.

- The identity

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-2}b + a^{n-1})$$

gives

$$b^n - a^n < (b - a)nb^{n-1}$$

if $0 < a < b$.

- Assume $y^n < x$. Choose $0 < h < 1$ so that

$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$

- Put $a = y$, $b = y + h$. Then

$$(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n.$$

- Thus $(y + h)^n < x$ and $y + h \in E$. Since $y + h > y$ this contradicts the fact that $y$ is an upper bound of $E$.

# Proof: 3/3. Case $y^n > x$.

- Assume that $y^n > x$ and set
$$k = \frac{y^n - x}{ny^{n-1}}.$$
- Then $0 < k < y$. If $t \geq y - k$ we conclude
$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$
- Thus $t^n > x$ and $t \notin E$. It follows that $y - k$ is an upper bound of $E$. But
$$y - k < y,$$
which contradicts the fact that $y$ is the least upper bound of $E$.
- Hence
$$y^n = x.$$

# Corollary

**Corollary**

If $a, b > 0$ are real numbers and $n \in \mathbb{N}$, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}$$

It is a consequence of the uniqueness property in the previous theorem.

# $x^y$ for $x, y \in \mathbb{R}$

Fix $b > 1$.

- If $m, n, p, q \in \mathbb{Z}$, $n, q > 0$ and $r = \frac{m}{n} = \frac{p}{q}$, then
$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$
- Hence, it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$.
- If $r, s \in \mathbb{Q}$, then
$$b^{r+s} = b^r b^s.$$
- If $x \in \mathbb{R}$ define
$$B(x) = \{b^t \ : \ t \in \mathbb{Q}, \ t \leq x\}.$$
- Then $b^n = \sup B(x)$ when $r \in \mathbb{Q}$. Hence, it makes sense to define
$$b^x = \sup B(x)$$
for every $x \in \mathbb{R}$.

# Decimals 1/2

Let $x > 0$ be real. Let $n_0$ be the largest integer such that $n_0 \leq x$.

**Remark**

Note that the existence of $n_0$ follows from the Archimedean property. Why?

Then, we define $n_1$ to be the largest integer such that

$$n_0 + \frac{n_1}{10} \leq x.$$

then, having $n_0, n_1$, we define $n_2$ to be the largest integer such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{100} \leq x.$$

We continue this procedure...

# Decimals 2/2

Having chosen

$$n_0, n_1, \ldots, n_{k-1}$$

let $n_k$ be the largest integer such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} \leq x.$$

Let

$$E = \{n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} \ : \ k \in \mathbb{N}_0\}.$$

Then one can show that $x = \sup E$.

# Decimal system - example

Example

Write down $0,25$ in the form $\frac{n}{m}$.

**Solution.** We write

$$0,25 = \frac{2}{10} + \frac{5}{100} = \frac{20}{100} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}.$$

□

# Decimal system - example

Example

Write down $x = 0,101010101\ldots$ in the form $\frac{n}{m}$.

**Solution.** Note that

$$10x = 10,10101010\ldots,$$

hence

$$10x = 10 + x$$

$$9x = 10 \iff x = \frac{10}{9}.$$

□

# The extended real number system

### The extended real number system

**The extended real number system** consists of real numbers $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$.

We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty$$

for all $x \in \mathbb{R}$.

### Example

If $E \subseteq \mathbb{R}$, $E \neq \emptyset$ but not bounded then

$$\sup E = +\infty.$$

# Properties of the extended real number system

## Properties

If $x \in \mathbb{R}$, then

1. $x + \infty = \infty$, $x - \infty = -\infty$, $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$,
2. if $x > 0$, then $x(+\infty) = +\infty$, $x(-\infty) = -\infty$,
3. if $x < 0$, then $x(+\infty) = -\infty$, $x(-\infty) = +\infty$.