Lesson 5

Dirichlet box principle, Cartesian products, relations and functions

MATH 311, Section 4, FALL 2022

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Integer and fractional part, absolute value 1/2

Let $x \in \mathbb{R}$.

Integer part

The integer part of x is

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \ : \ n \leq x\}.$$

Fractional part

The fractional part of x is

$$\{x\} = x - \lfloor x \rfloor.$$

Integer and fractional part, absolute value 2/2

Absolute value

The absolute value of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Example 1

If $x = 2.5$, then $\lfloor x \rfloor = 2$, $\{x\} = 0.5$, $|x| = 2.5$.

Example 2

If $x = -3.3$, then $\lfloor x \rfloor = -4$, $\{x\} = 0.7$, $|x| = 3.3$.

Propoerties of $|x|$

Theorem

For $x, y \in \mathbb{R}$ one has

- $|x| = \sqrt{x^2}$,
- $|xy| = |x||y|$,
- $x \leq |x|$ and $x \geq -|x|$
- $|x + y| \leq |x| + |y|$, (triangle inequality).
- $||x| - |y|| \leq |x - y| \leq |x| + |y|$, (triangle inequality).

Proof:

- Note that $|x|^2 = x^2$ and the equation $z^2 = x^2$ has a unique solution for $z > 0$. Since both $z = |x|$ and $z = \sqrt{x^2}$ solve this equation we must have
$$|x| = \sqrt{x^2}.$$

Proof

- Observe that $|xy| = \sqrt{(xy)^2} = \sqrt{x^2y^2} = \sqrt{x^2}\sqrt{y^2} = |x||y|$.
- Clearly $x \leq |x|$ for all $x \in \mathbb{R}$. Similarly, $-x \leq |x|$ giving $x \geq -|x|$.
- Since $x \leq |x|$ and $y \leq |y|$, then $x + y \leq |x| + |y|$. We also have $-(|x| + |y|) \leq x + y$, since $-|x| \leq x$ and $-|y| \leq y$. Hence
$$-(|x| + |y|) \leq x + y \leq |x| + |y| \qquad \Longleftrightarrow \qquad |x + y| \leq |x| + |y|$$
- Note that
$$|x| = |y + x - y| \leq |y| + |x - y|, \quad \text{and}$$
$$|y| = |x + y - x| \leq |x| + |x - y|.$$
Thus
$$-|x - y| \leq |x| - |y| \quad \text{and} \quad |x| - |y| \leq |x - y|,$$
which gives
$$||x| - |y|| \leq |x - y|.$$
□

Exercise

Two $a, b \in \mathbb{R}$ are equal iff for every $\varepsilon > 0$ it follows

$$|a - b| < \varepsilon.$$

Proof (⇐). If $a = b$, then $|a - b| = 0 < \varepsilon$ for any $\varepsilon > 0$.

Proof (⇒). Suppose that for any $\varepsilon > 0$ one has $|a - b| < \varepsilon$. If $a = b$, then we are done. Assume that $a \neq b$ and take $\varepsilon_0 = |a - b| > 0$. Taking any $0 < \varepsilon < \varepsilon_0$ one has

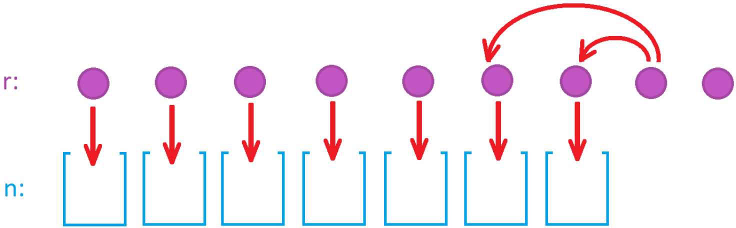
$$0 < \varepsilon_0 = |a - b| < \varepsilon < \varepsilon_0,$$

which is impossible. □

Pigeonhole principle

Dirichlet's box principle

If r objects are placed in n boxes and $r > n$, then at least one of the boxes contains more than one object.



Dirichlet principle

Theorem (Dirichlet)

Let α, Q be real numbers, $Q \geq 1$. There exist $a, q \in \mathbb{Z}$ such that $1 \leq q \leq Q$ and a, q are relatively prime such that

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{qQ} \leq \frac{1}{q^2}.$$

Proof. Let $N = \lfloor Q \rfloor$. We will consider three cases.

- Case 1. $\{\alpha q\} \in \left[0, \frac{1}{N+1}\right)$,
- Case 2. $\{\alpha q\} \in \left[\frac{N}{N+1}, 1\right)$,
- Case 3. $\{\alpha q\} \notin \left[0, \frac{1}{N+1}\right) \cup \left[\frac{N}{N+1}, 1\right)$.

Proof: Case 1.

- Suppose that
$$\{\alpha q\} \in \left[0, \frac{1}{N+1}\right)$$
for some positive integer $q \leq N$.
- If $a = \lfloor \alpha q \rfloor$, then
$$0 \leq \{\alpha q\} = \alpha q - a < \frac{1}{N+1},$$
so (dividing both sides by q):
$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{(N+1)q} < \frac{1}{qQ} < \frac{1}{q^2}.$$

Proof: Case 2.

- Suppose that

$$\{\alpha q\} \in \left[\frac{N}{N+1}, 1\right)$$

for some positive integer $q \leq N$.

- If $a = \lfloor \alpha q \rfloor + 1$, then

$$\frac{N}{N+1} \leq \{\alpha q\} = \alpha q - Q + 1 \leq 1,$$

implies

$$|\alpha q - a| < \frac{1}{N+1},$$

so (dividing both sides by q):

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{(N+1)q} < \frac{1}{qQ} < \frac{1}{q^2}.$$

Proof: Case 3. 1/2

Suppose that

$$\{\alpha q\} \in \left[\frac{1}{N+1}, \frac{N}{N+1}\right).$$

for all $1 \le q \le N$. Then each of the N numbers

$$\{\alpha\}, \{2\alpha\}, \{3\alpha\}, \dots, \{N\alpha\}$$

lies in $N-1$ intervals

$$\left[\frac{1}{N+1}, \frac{2}{N+1}\right), \left[\frac{2}{N+1}, \frac{3}{N+1}\right), \left[\frac{3}{N+1}, \frac{4}{N+1}\right), \dots, \left[\frac{N-1}{N+1}, \frac{N}{N+1}\right)$$

Therefore, by the Dirichlet's box principle there exist $1 \le j \le N-1$ and $q_1, q_2 \in \{1, 2, \dots, N\}$, $q_1 < q_2$, such that

$$q_1, q_2 \in \left[\frac{j}{N+1}, \frac{j+1}{N+1}\right).$$

Proof: Case 3. 2/2

Let $q = q_2 - q_1$ and

$$a = \lfloor \alpha q_2 \rfloor - \lfloor \alpha q_1 \rfloor.$$

Then

$$\begin{aligned} |\alpha q - a| &= |(\alpha q_2 - \lfloor \alpha q_2 \rfloor) - (\alpha q_1 - \lfloor \alpha q_1 \rfloor)| \\ &= |\{\alpha q_2\} - \{\alpha q_1\}| < \frac{1}{N+1}. \end{aligned}$$

Thus

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{(N+1)q} < \frac{1}{qQ} < \frac{1}{q^2}.$$

The proof is finished. □

Ordered pairs

Ordered pairs

The ordered pair (x, y) is precisely the set $\{\{x\}, \{x, y\}\}$.

Theorem

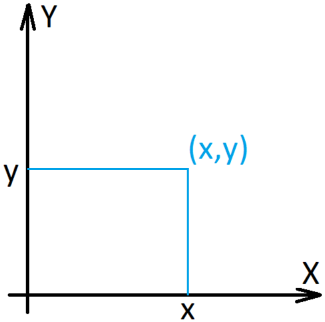
$(x, y) = (u, v)$ iff $x = u$ and $y = v$.

Cartesian products

Cartesian products

If X and Y are sets, their **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

$$X \times Y = \{(x, y) \,:\, x \in X, y \in Y\}$$



Cartesian products - examples

Example 1

If $X = \{1, 2, 3\}$, $Y = \{4, 5\}$, then

$$X \times Y = \{(1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5)\}.$$

Example 2

If $X = \{1, 2\}$, $Y = \{1, 2\}$, then

$$X \times Y = \{(1, 1), (1, 2), (2, 1), (2, 2)\}.$$

Example 3

If $X \neq \emptyset$ and $Y = \emptyset$, then $X \times Y = \emptyset$.

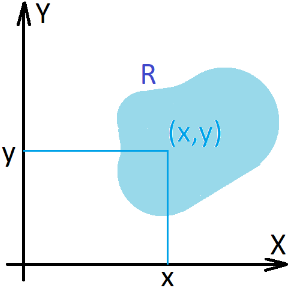
Relations

Relations

A relation from X to Y is a subset R of $X \times Y$, i.e. $R \subseteq X \times Y$.

If $X = Y$ we speak about relations on X.

If R is a relation from X to Y we shall sometimes write xRy to mean that $(x, y) \in R \subseteq X \times Y$.



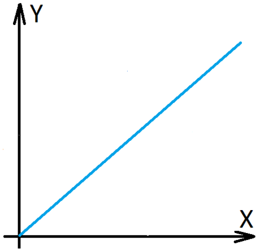
Relations - examples

Example 1

$$xRy \iff x = y$$

This relation corresponds to the diagonal Δ in $X \times X$:

$$\Delta = \{(x, x) \; : \; x \in X\} \subseteq X \times X.$$



Now we present more examples of relations.

Equivalence relations

Equivalence relations

An equivalence relation is a relation on X such that:

1. xRx for all $x \in X$, (reflexivity).
2. xRy iff yRx for all $x, y \in X$, (symmetry).
3. if xRy and yRz, then xRz for all $x, y, z \in R$. (transitivity).

Equivalence classes

An equivalence class of an element $x \in X$ is the set $\{y \in X : xRy\}$.

X is the disjoint union of the equivalence classes.

Equivalence relations - examples 1/2

Example

Let $X = \mathbb{Z}$. Consider

$$xRy \iff x \equiv y \mod 5 \iff 5|(x-y).$$

the equivalence classes corresponding to the relation R are the sets:

$$E_0 = \{5k \,:\, k \in \mathbb{Z}\},$$

$$E_1 = \{5k+1 \,:\, k \in \mathbb{Z}\},$$

$$E_2 = \{5k+2 \,:\, k \in \mathbb{Z}\},$$

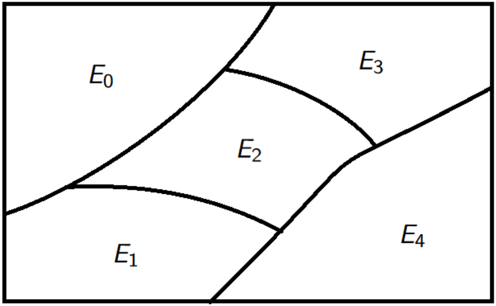
$$E_3 = \{5k+3 \,:\, k \in \mathbb{Z}\},$$

$$E_4 = \{5k+4 \,:\, k \in \mathbb{Z}\}.$$

Equivalence relations - examples 2/2

We have

$$\mathbb{Z} = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4.$$



Example 2

Orderings are also relations (will be discussed later).

Functions

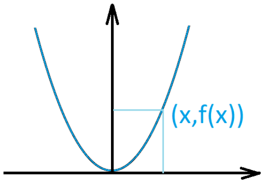
Functions

A function $f : X \to Y$ is a relation from X to Y with the property that for every $x \in X$ there is a unique element $y \in Y$ such that xRy in which case we write

$$y = f(x).$$

Example 1

$X = \mathbb{R}$, $f(x) = x^2$.

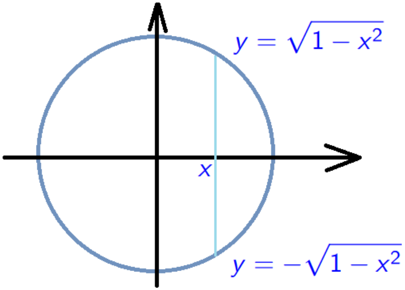


A relation which is not a function

Example 2

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}.$$

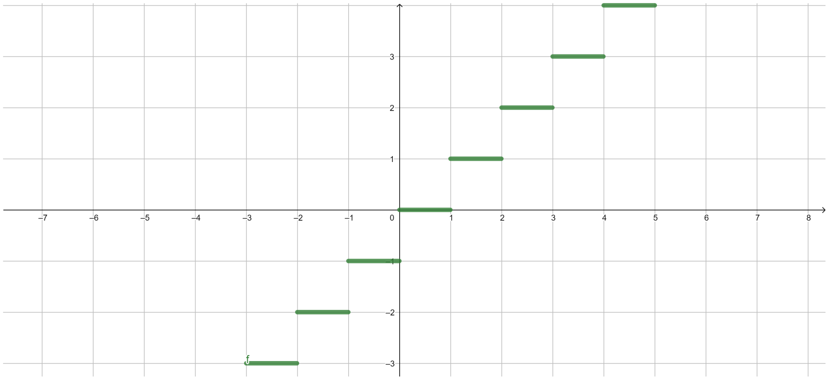
This is not a function!



Examples of functions - integer part

Integer part

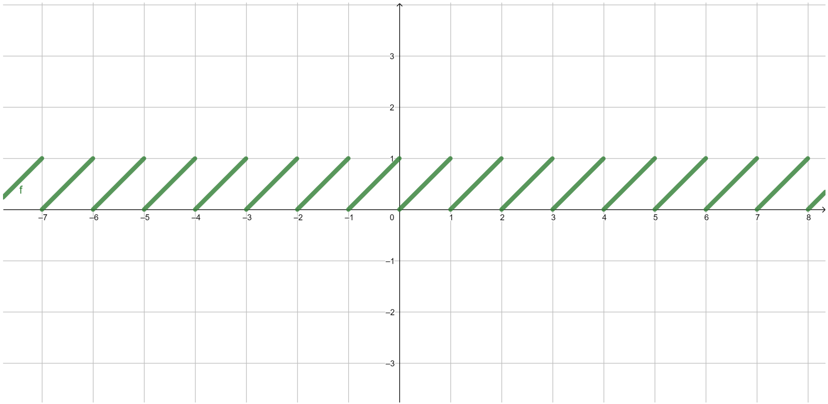
$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}.$$



Examples of functions - fractional part

Fractional part

$$\{x\} = x - \lfloor x \rfloor.$$



Examples of functions - absolute value

Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}.$$