Lesson 6

Functions and their properties
Trigonometric functions $\sin(\theta)$ and $\cos(\theta)$

MATH 311, Section 4, FALL 2022

September 23, 2022

Functions

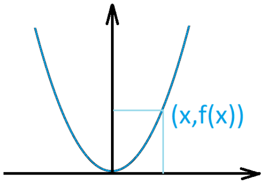
Functions

A function $f : X \to Y$ is a relation from X to Y with the property that for every $x \in X$ there is a unique element $y \in Y$ such that xRy in which case we write

$$y = f(x).$$

Example 1

$X = \mathbb{R}$, $f(x) = x^2$.

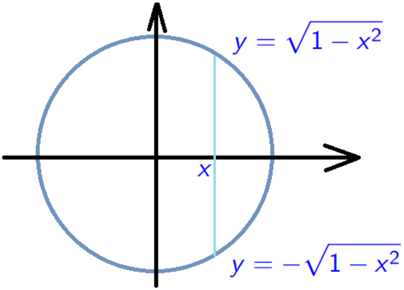


A relation which is not a function

Example 2

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}.$$

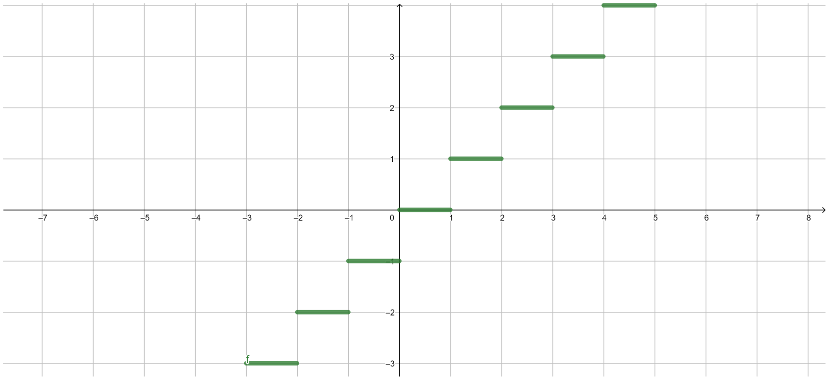
This is not a function!



Examples of functions - integer part

Integer part

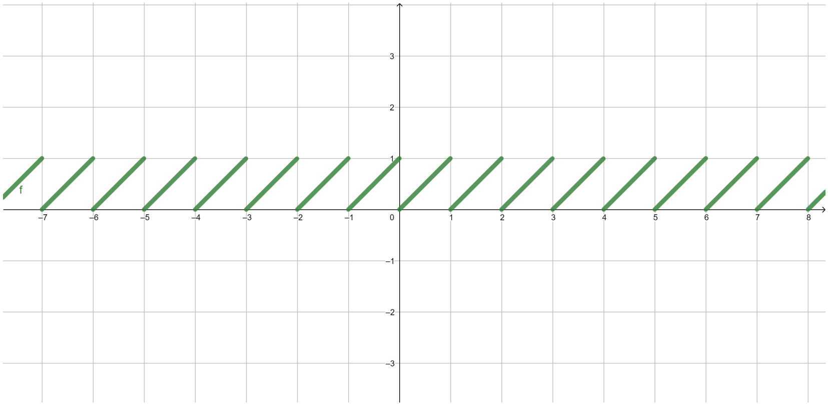
$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}.$$



Examples of functions - fractional part

Fractional part

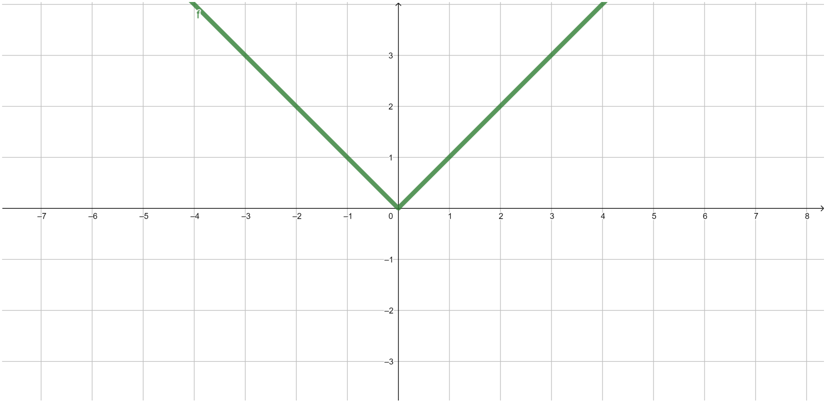
$$\{x\} = x - \lfloor x \rfloor.$$



Examples of functions - absolute value

Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}.$$



Composition of functions

Composition of functions

If $f : X \to Y$ and $g : Y \to Z$ are functions we define their **composition** $g \circ f : X \to Z$ by setting

$$g \circ f(x) = g(f(x)) \quad \text{for} \quad x \in X.$$

Example

If $f(x) = x^2 - 2$ and $g(x) = |x|$, then $g \circ f(x) = |x^2 - 2|$.

Composition of the functions - example

$$f(x) = x^2 - 2, \qquad g(x) = |x|,$$
$$g \circ f(x) = |x^2 - 2|$$

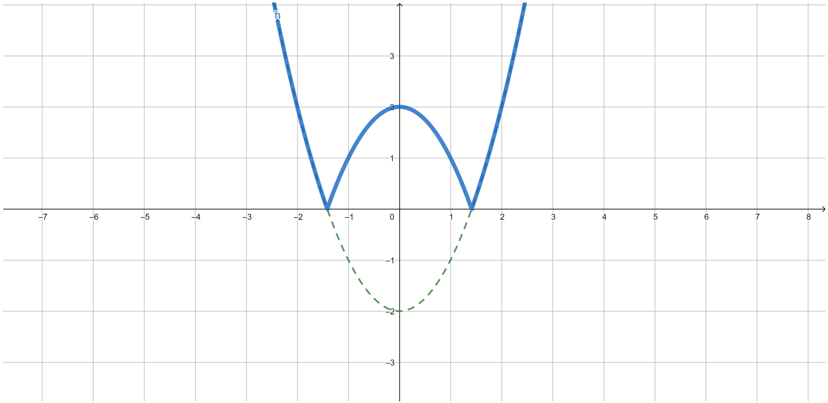


Image and inverse image

Image

If $D \subseteq X$, we define **the image** of D under the function $f : X \to Y$ by

$$f[D] = \{f(x) \ : \ x \in D\}.$$

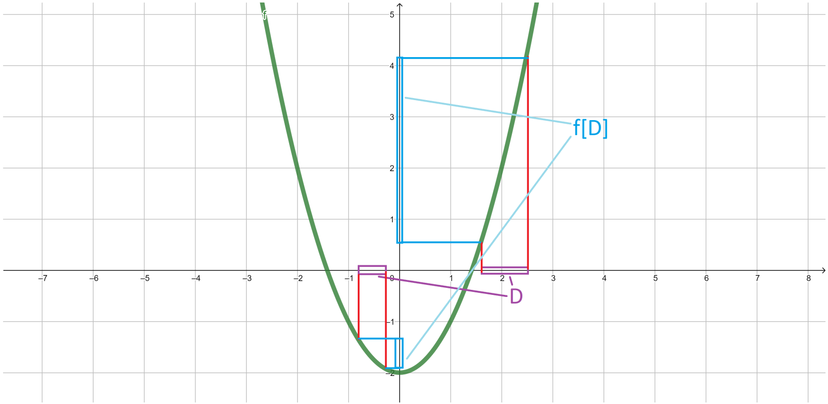
Inverse image

If $E \subseteq Y$, we define **the inverse image** of E under the function $f : X \to Y$ by

$$f^{-1}[E] = \{x \in X \ : \ f(x) \in E\}.$$

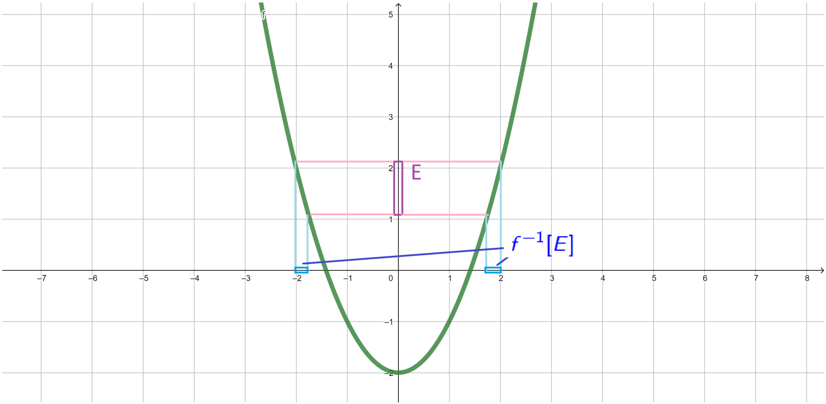
Image - example

$$f[D] = \{f(x) \ : \ x \in D\}.$$



Inverse image - example

$$f^{-1}[E] = \{x \in X \ : \ f(x) \in E\}.$$



Inverse image - properties

For every function $f : X \to Y$ one has

1. $$f^{-1}\left[\bigcup_{\alpha \in A} E_\alpha\right] = \bigcup_{\alpha \in A} f^{-1}[E_\alpha],$$

2. $$f^{-1}\left[\bigcap_{\alpha \in A} E_\alpha\right] = \bigcap_{\alpha \in A} f^{-1}[E_\alpha],$$

3. $$f^{-1}[E^c] = (f^{-1}[E])^c.$$

Image - properties

For every function $f : X \to Y$ one has

$$f\left[\bigcup_{\alpha\in A} E_\alpha\right] = \bigcup_{\alpha\in A} f[E_\alpha].$$

Exercise

Corresponding formulas for intersection and complements may not be true.

Domain and range

Domain

If $f : X \to Y$ is a function, X is called **the domain** of f and denoted by

$$\mathrm{dom}(f) = X.$$

Range

If $f : X \to Y$ is a function, $f[X]$ is called **the range** of f denoted by

$$\mathrm{rgn}(f) = f[X].$$

Example

If $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2$, then

$$\mathrm{dom}(f) = \mathbb{R} \qquad \text{and} \qquad \mathrm{rgn}(f) = [0, \infty).$$

Injective functions, 1/2

Injective functions

The function f is said to be **injective** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 1

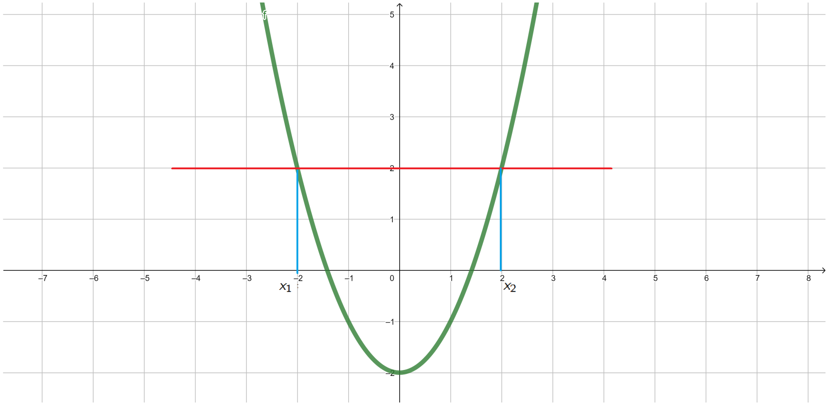
$f(x) = x^2 - 2$ is not injective since for $x_1 = -2$ and $x_2 = 2$ we have

$$f(x_1) = f(-2) = (-2)^2 - 2 = 2,$$

$$f(x_2) = f(2) = 2^2 - 2 = 2.$$

Injective functions - example 1/2

$f(x) = x^2 - 2$



Injective functions, 2/2

Example 2

$f(x) = x^3 + 4$ is injective.

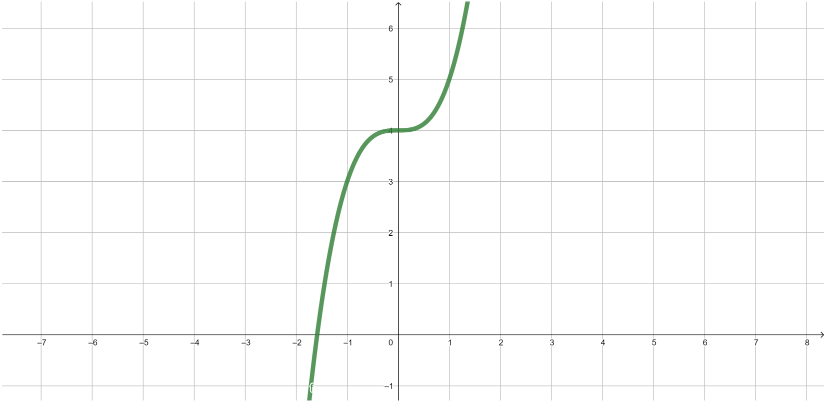
Proof: Indeed, suppose that $f(x_1) = f(x_2)$, then

$$\begin{aligned} f(x_1) = f(x_2) &\iff \\ x_1^3 + 4 = x_2^3 + 4 &\iff \\ x_1^3 = x_2^3 &\iff x_1 = x_2. \end{aligned}$$

□

Injective functions - example 2/2

$f(x) = x^3 + 4$



Surjective functions

Surjective functions

$f : X \to Y$ is said to be **surjective** if $f[X] = Y$.

Example 1

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 5$ is surjective.

Example 2

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 2$ is not surjective since

$$f[\mathbb{R}] = [-2, +\infty) \neq \mathbb{R}.$$

Example 3

Every mapping $f : X \to Y$ is surjective if $Y = f[X]$.

Bijective functions

Bijective functions

$f : X \to Y$ is **bijective** if it is both injective and surjective.

Example 1

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ is bijective is $a \neq 0$.

Example 2

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 5$ is bijective.

Example 3

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ is not bijective since it is not injective.

Inverse functions

Inverse functions

If $f : X \to Y$ is bijective it has an inverse $f^{-1} : Y \to X$ such that

$$f^{-1} \circ f \quad \text{and} \quad f \circ f^{-1}$$

are both identity functions, i.e.

$$f^{-1} \circ f(x) = x \quad \text{and} \quad f \circ f^{-1}(y) = y \quad \text{for all} \quad x \in X, y \in Y.$$

Example 1

If $a \neq 0$, then $f(x) = ax + b$ has an inverse given by

$$f^{-1}(x) = \frac{x - b}{a}.$$

Restriction of the function

Restriction

If $A \subseteq X$ we denote $f_{|A}$ **the restriction** of $f : X \to Y$ to A:

$$f_{|A} : A \to Y, \qquad f_{|A}(x) = f(x) \qquad \text{for all} \qquad x \in A.$$

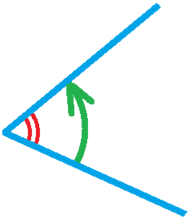
Example 1

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Let $A = [0, +\infty)$ and let $g(x) = f_{|A}$. Then f is not injective, but g is injective.

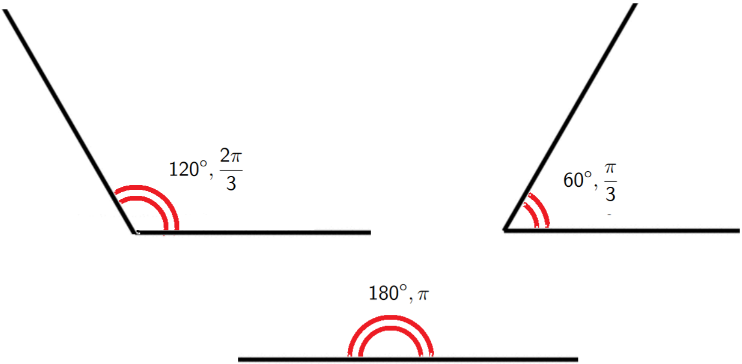
Angle

An angle can be thought of as the amount of rotation required to take one straight line to another line with a common endpoint. Angles can be measured in degrees or in radians (abbreviated as rad). The angle given by a complete revolution contains 360° and

$$2\pi \text{ rad} = 360^\circ$$



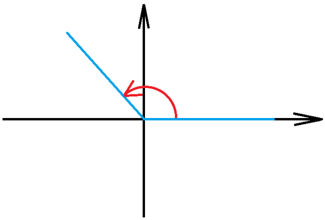
Examples



Degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$180°$	$270°$	$360°$
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π	$\frac{3\pi}{2}$	2π

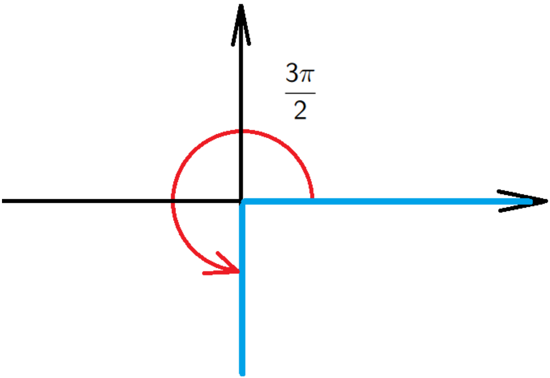
Angle in standard position

An angle is in standard position if its vertex (the endpoint of two rays) is located at the origin and one ray is on the positive x-axis. The ray on the x-axis is called the **initial side** and the other ray is called the **terminal side**.

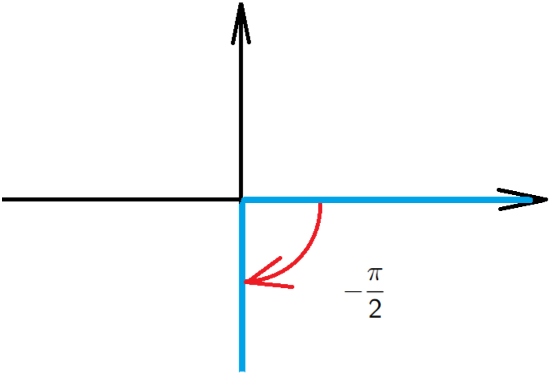


Using this terminology the angle is measured by the amount of rotation from the initial side to the terminal side. If measured in a counterclockwise direction the measurement is **positive**. If measured in a clockwise direction the measurement is **negative**.

Example 1



Example 2

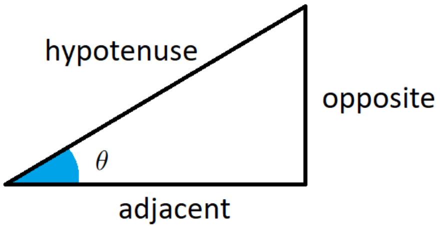


$\sin(\theta)$ and $\cos(\theta)$ for acute angle

$\sin(\theta)$ and $\cos(\theta)$ for acute angle

For an acute angle θ the quantities $\sin(\theta)$ and $\cos(\theta)$ are defined as ratios of lengths of sides of a right triangle as follows

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}.$$

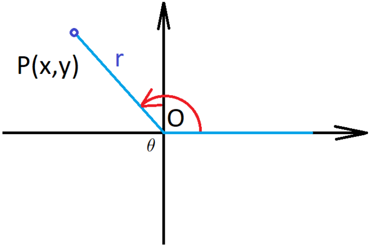


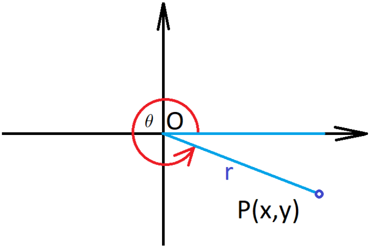
$\sin(\theta)$ and $\cos(\theta)$ for any angle

$\sin(\theta)$ and $\cos(\theta)$ for any angle

The previous definition does not apply to angles which are not acute, so for a general angle in standard position we let $P(x, y)$ be any point on the terminal side and we let r be the distance $|OP|$ (O is the origin). Then we define

$$\sin(\theta) = \frac{y}{r}, \qquad \cos(\theta) = \frac{x}{r}.$$



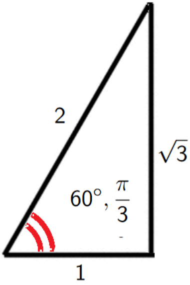


Examples 1

Example 1/4

For an acute angle $\theta = \frac{\pi}{3}$ we have

$$\sin(\theta) = \frac{\sqrt{3}}{2}, \qquad \cos(\theta) = \frac{1}{2}.$$

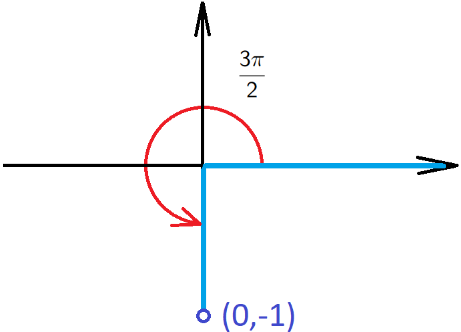


Examples 2/4

Example 2

For an angle $\theta = \frac{3\pi}{2}$ we have a point $(0, -1)$ on the terminal side, so

$$\sin(\theta) = \frac{-1}{1} = -1, \qquad \cos(\theta) = \frac{0}{1} = 0.$$

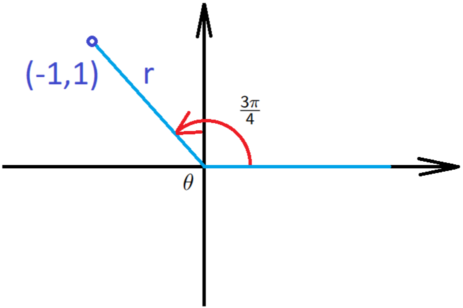


Examples 3/4

Example 3

For an angle $\theta = \frac{3\pi}{4}$ we have a point $(-1, 1)$ on the terminal side, so

$$\sin(\theta) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \cos(\theta) = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

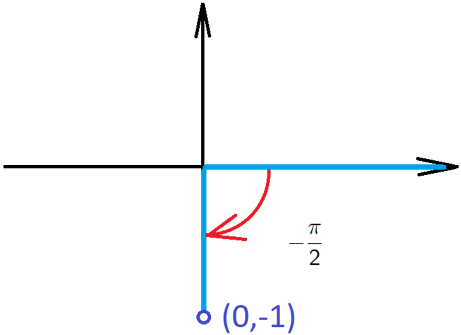


Examples 4/4

Example 4

For an angle $\theta = -\frac{\pi}{2}$ we have a point $(0, -1)$ on the terminal side, so

$$\sin(\theta) = \frac{-1}{1} = -1, \qquad \cos(\theta) = \frac{0}{1} = 0.$$



Values of trigonometric functions

Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0

Trigonometric identities

Pythagorean identity

For any angle θ we have

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

Proof. Let $P(x, y)$ be any point on the terminal side. By the definition of $\sin(\theta)$ and $\cos(\theta)$ we have

$$\sin^2(\theta) + \cos^2(\theta) = \frac{y^2}{|OP|^2} + \frac{x^2}{|OP|^2}.$$

But by the Pythagorean theorem

$$|OP|^2 = x^2 + y^2,$$

hence

$$\sin^2(\theta) + \cos^2(\theta) = \frac{|OP|^2}{|OP|^2} = 1. \qquad \square$$

Trigonometric identities

Trigonometric identities

We have the following identities, which can be observed geometrically:

a. $\sin(x+\frac{\pi}{2}) = \cos(x)$,
b. $\cos(x+\frac{\pi}{2}) = -\sin(x)$,
c. $\sin(x+\pi) = -\sin(x)$,
d. $\cos(x+\pi) = -\cos(x)$,
e. $\sin(x+2\pi) = \sin(x)$,
f. $\cos(x+2\pi) = \cos(x)$.

Graph of $\sin(x)$ and $\cos(x)$