Lesson 7

Axiom of Choice, Cardinality, Cantor's theorem

MATH 311, Section 4, FALL 2022

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Set of maps

Cartesian product

If $(X_\alpha)_{\alpha\in A}$ is an indexed family of sets, their Cartesian product

$$\prod_{\alpha\in A} X_\alpha$$

is the set of all maps $f : A \to \bigcup_{\alpha\in A} X_\alpha$ so that $f(\alpha)\in X_\alpha$ for all $\alpha\in A$.

Projection map

If $X = \prod_{\alpha\in A} X_\alpha$ and $\alpha\in A$ we define α-**th projection** or **coordinate map** $\pi_\alpha : X \to X_\alpha$ by $\pi_\alpha(f) = f(\alpha)$.

We will also write $x = (x_\alpha)_{\alpha\in A} \in X = \prod_{\alpha\in A} X_\alpha$ instead of f, and x_α instead of $f(\alpha)$.

Set of maps - examples 1/2

Example 1

If $A = \{1, 2, \ldots, n\}$, then

$$X = \prod_{j=1}^{n} X_j = X_1 \times X_2 \times \ldots \times X_n$$
$$= \{(x_1, x_2, \ldots, x_n) \ : \ x_j \in X_j \text{ for all } j = 1, 2, \ldots, n\}.$$

Example 2

If $A = \mathbb{N}$, then

$$X = \prod_{j=1}^{\infty} X_j = X_1 \times X_2 \times \ldots = \{(x_1, x_2, \ldots) \ : \ x_j \in X_j \text{ for all } j = 1, 2, \ldots\}.$$

Set of maps - examples 2/2

Example 3

If the sets X_α are all equal to some fixed set Y, we denote

$$X = \prod_{\alpha \in A} X_\alpha = Y^A.$$

Y^A-the set of all mappings from A to Y.

Example 4

$\mathbb{Z}^\mathbb{N}$ - the set of all sequences of integers.

$\mathbb{R}^\mathbb{N}$ - the set of all sequences of real numbers.

The axiom of choice

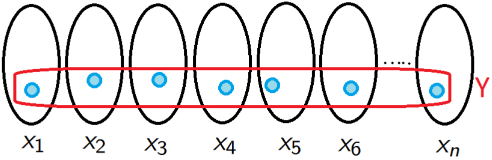
The axiom of choice

If $(X_\alpha)_{\alpha \in A}$ is a nonempty collection of nonempty sets, then

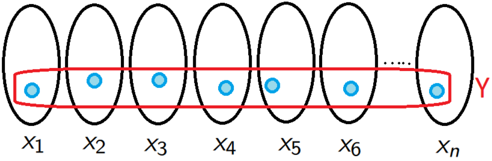
$$\prod_{\alpha \in A} X_\alpha \neq \emptyset.$$

Corollary

If $(X_\alpha)_{\alpha \in A}$ is a disjoint collection of nonempty sets there is a set (called **the selector**) $Y \subseteq \bigcup_{\alpha \in A} X_\alpha$ such that $Y \cap X_\alpha$ contains precisely one element for each $\alpha \in A$.



Proof of Corollary



Take $f \in \prod_{\alpha \in A} X_\alpha \neq \emptyset$. Define

$$Y = f[A],$$

then

$$Y \cap X_\alpha = \{f(\alpha)\},$$

since $f(\alpha) \in X_\alpha$. □

Cardinality

Cardinality

If X and Y are nonempty sets, we define the expressions

$$\mathrm{card}\,(X) \leq \mathrm{card}\,(Y) \quad \text{(injective)},$$

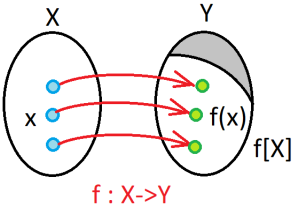
$$\mathrm{card}\,(X) = \mathrm{card}\,(Y) \quad \text{(bijective)},$$

$$\mathrm{card}\,(X) \geq \mathrm{card}\,(Y) \quad \text{(surjective)},$$

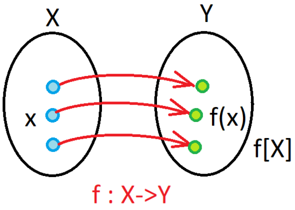
to mean that there exists $f : X \to Y$ which is injective, bijective, surjective respectively.

Cardinality - pictures 1/2

$\mathrm{card}\,(X) \leq \mathrm{card}\,(Y)$, (injective),

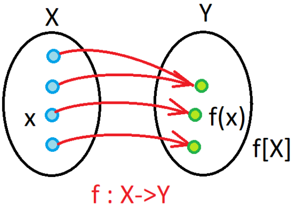


$\mathrm{card}\,(X) = \mathrm{card}\,(Y)$, (bijective)



Cardinality - pictures 2/2

$\mathrm{card}\,(X) \geq \mathrm{card}\,(Y)$, (surjective)



- We also define $\mathrm{card}\,(X) < \mathrm{card}\,(Y)$ to mean that there is an injection but not a bijection.
- We also have $\mathrm{card}\,(\emptyset) < \mathrm{card}\,(X)$ and $\mathrm{card}\,(X) > \mathrm{card}\,(\emptyset)$ for all $X \neq \emptyset$.

$\operatorname{card}(X)$ -example

Example

Let

$$X = \{1, 2, 3, 4, \ldots\}$$

$$Y = \{101, 102, 103, \ldots, \}.$$

Prove that $\operatorname{card}(X) = \operatorname{card}(Y)$.

Solution. Let us define $f : X \to Y$ by

$$f(x) = x + 100,$$

then f is a bijection between X and Y, so $\operatorname{card}(X) = \operatorname{card}(Y)$. $\square$

$\mathrm{card}\,(X)$ - example

Example

Let

$$X = \{1, 2, 3, 4, \ldots\}$$

$$Y = \{1^2, 2^2, 3^2, 4^2, \ldots, \}.$$

Prove that $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

Solution 1. Let us define $f : X \to Y$ by

$$f(x) = x^2,$$

then f is a bijection between X and Y, so $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □

Solution 2. Let us define $g : Y \to X$ by

$$f(x) = \sqrt{x},$$

then g is a bijection between X and Y, so $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □

$\mathrm{card}\,(X)$ - example

Example

Let

$$X = \{1, 2, 3\}$$

$$Y = \{2, 4, 6, 8\}.$$

Prove that $\mathrm{card}\,(X) < \mathrm{card}\,(Y)$.

Solution. Note that $f(x) = 2x$ is an injection from X to Y, so $\mathrm{card}\,(X) \leq \mathrm{card}\,(Y)$. On the other hand, any function from X to Y takes at most 3 values, so it is not a surjection, so $\mathrm{card}\,(X) < \mathrm{card}\,(Y)$. □

$\mathrm{card}\,(X)$ - example

Example

Let

$$X = [0, 1]$$
$$Y = [1, 3].$$

Prove that $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

Solution. Let us define $f : X \to Y$ by

$$f(x) = 2x + 1,$$

then f is a bijection between X and Y, so $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □

Proposition

Proposition

We have

$$\mathrm{card}\,(X) \le \mathrm{card}\,(Y) \iff \mathrm{card}\,(Y) \ge \mathrm{card}\,(X).$$

Proof (⇒). Assume that $\mathrm{card}\,(X) \le \mathrm{card}\,(Y)$. This means that there is an injection $f : X \to Y$. Thus f is a bijection $f : X \to f[X] \subseteq Y$. Let f^{-1} be the inverse $f^{-1} : f[X] \to X$. Pick $x_0 \in X$ and define g by

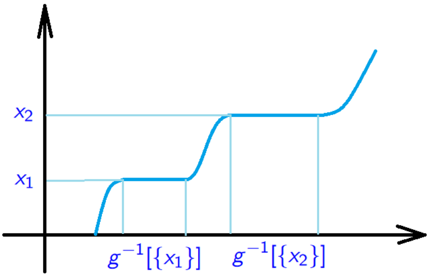
$$g(y) = \begin{cases} f^{-1}(y) \text{ if } y \in f[X], \\ g(y) = x_0 \text{ if } y \in Y \setminus f[X]. \end{cases}$$

Then we see that g is surjective from Y to X.

Proof: 1/2

Proof (⇐). If $\mathrm{card}\,(Y) \geq \mathrm{card}\,(X)$, then there is a surjection $g : Y \to X$. Then $g[Y] = X$, and, consequently, $g^{-1}[\{x\}]$ are nonempty and

$$g^{-1}[\{x_1\}] \cap g^{-1}[\{x_2\}] = \emptyset \quad \text{if} \quad x_1 \neq x_2.$$



Proof: 2/2

Using the axiom of choice the set $\prod_{x \in X} g^{-1}[\{x\}] \neq \emptyset$. Taking

$$f \in \prod_{x \in X} g^{-1}[\{x\}]$$

we see that f is an injection from X to Y. Indeed, if $x_1 \neq x_2$, then $f(x_1) \in g^{-1}[\{x_1\}]$ and $f(x_2) \in g^{-1}[\{x_2\}]$, but

$$g^{-1}[\{x_1\}] \cap g^{-1}[\{x_2\}] = \emptyset,$$

thus $f(x_1) \neq f(x_2)$. □

$\text{card}(X)$ - example

Example

Let

$$X = [0, 1],$$

$$Y = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\}.$$

Prove that $\text{card}(X) \geq \text{card}(Y)$.

Solution. One has $\text{card}(Y) \leq \text{card}(X)$. Indeed, define $f : Y \to X$ by

$$f(x) = x,$$

which is injective, so $\text{card}(X) \geq \text{card}(Y)$. □

Remark

Actually we have $\text{card}(X) > \text{card}(Y)$, which will be proved later.

Banach lemma

Theorem

For any sets X and Y

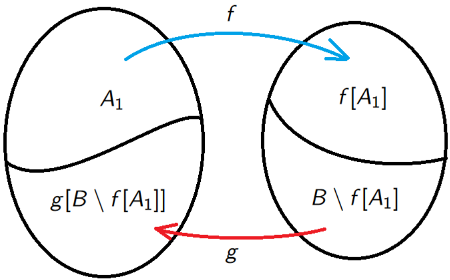
$$\text{either} \quad \mathrm{card}\,(X) \leq \mathrm{card}\,(Y) \quad \text{or} \quad \mathrm{card}\,(Y) \leq \mathrm{card}\,(X).$$

Banach lemma

Let $f : A \to B$ and $g : B \to A$ be injections. Then there are sets A_1, A_2, B_1, B_2 such that

1. $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$,
2. $B_1 \cup B_2 = B$ and $B_1 \cap B_2 = \emptyset$,
3. $f[A_1] = B_1$ and $f[A_2] = B_2$.

Banach lemma - picture



Proof of Banach lemma 1/2

Consider the map $\Phi : P(A) \to P(A)$ defined by

$$\Phi(X) = A \setminus g[B \setminus f[X]] \quad \text{for} \quad X \in P(X).$$

Since g is injective we note that

$$\Phi\left[\bigcup_{t \in I} A_t\right] = \bigcup_{t \in I} \Phi[A_t]$$

for any family $(A_t)_{t \in I}$ such that $A_t \subseteq A$ for all $t \in I$. Consider

$$\Omega = \emptyset \cup \Phi[\emptyset] \cup \Phi \circ \Phi[\emptyset] \cup \ldots \cup \overbrace{\Phi \circ \Phi \circ \ldots \circ \Phi}^{n}[\emptyset] \cup \ldots,$$

in other words

$$\Omega = \bigcup_{n=0}^{\infty} \Phi^n[\emptyset].$$

Proof of Banach lemma

Then

$$\Phi[\Omega] = \bigcup_{n=0}^{\infty} \Phi^{n+1}[\emptyset] = \bigcup_{n=1}^{\infty} \Phi^{n}[\emptyset] = \left(\bigcup_{n=1}^{\infty} \Phi^{n}[\emptyset]\right) \cup \emptyset = \bigcup_{n=0}^{\infty} \Phi^{n}[\emptyset] = \Omega,$$

thus the set Ω if a fixed point of Φ, i.e. $\Phi[\Omega] = \Omega$. Taking

$$A_1 = \Omega, \quad A_2 = A \setminus A_1,$$

$$B_1 = f[A_1], \quad B_2 = B \setminus B_1$$

we obtain

$$A_1 = \Phi[A_1] = A \setminus g[B \setminus f[A_1]] = A \setminus g[B \setminus B_1] = A \setminus g[B_2],$$

thus

$$A_2 = A \setminus A_1 = A \setminus (A \setminus g[B_2]) = g[B_2].$$

Cantor–Bernstein–Schröder theorem (1.5.11)

Cantor–Bernstein–Schröder theorem

If $\mathrm{card}\,(X) \leq \mathrm{card}\,(Y)$ and $\mathrm{card}\,(Y) \leq \mathrm{card}\,(X)$, then $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

Proof. By Banach lemma there are A_1, A_2, B_1, B_2 such that

$$A_1 \cup A_2 = X, \quad f[A_1] = B_1, \quad A_1 \cap A_2 = \emptyset$$

$$B_1 \cup B_2 = X, \quad g[B_2] = A_2, \quad B_1 \cap B_2 = \emptyset$$

whenever $f : X \to Y$ and $g : Y \to X$ are injections. Define $h : X \to Y$ by setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_1, \\ g^{-1}(x) & \text{if } x \in A_2. \end{cases}$$

Then we see that h is a bijection between X and Y. □

Cantor–Bernstein–Schröder theorem - example

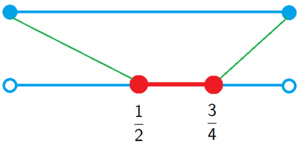
Example

Let $X = [0, 1]$, $Y = (0, 1)$. Prove that $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

Solution. Note that $\mathrm{card}\,(Y) \le \mathrm{card}\,(X)$, because $f(x) = x$ is injective. Then, let us prove that $\mathrm{card}\,(X) \le \mathrm{card}\,(Y)$. We define

$$g(x) = \frac{x}{4} + \frac{1}{2}.$$

One can check that g is injective, so, by **Cantor–Bernstein–Schröder theorem**, $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □



Cantor's theorem

Theorem (Cantor, Theorem 1.6.2)

For any set X we have $\text{card}\,(X) < \text{card}\,(P(X))$.

Proof. The map $f : X \to P(X)$ defined by $f(x) = \{x\}$ is an injection from X to $P(X)$. Thus $\text{card}\,(X) \leq \text{card}\,(P(X))$.

We now show that there is no bijection between X and $P(X)$. Let $g : X \to P(X)$ and consider the set

$$Y = \{x \in X \ : \ x \notin g(x)\} \in P(X).$$

Then we claim that $Y \notin g[X]$.

Proof

If

$$Y = \{x \in X \ : \ x \notin g(x)\} \in g[X],$$

then there is $x_0 \in X$ so that $g(x_0) = Y$.

- On the one hand
$$x_0 \in Y \iff x_0 \notin g(x_0).$$
- On the other hand
$$x_0 \in Y \iff x_0 \in g(x_0).$$
- Thus
$$x_0 \in g(x_0) \iff x_0 \notin g(x_0),$$
which is impossible, and we conclude $\mathrm{card}\,(X) < \mathrm{card}\,(P(X))$. □