Lesson 8

Countable sets, cardinality continuum

MATH 311, Section 4, FALL 2022

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Countable set

Countable set

A set X is called **countable (or denumerable)** if

$$\mathrm{card}\,(X) \le \mathrm{card}\,(\mathbb{N}).$$

Example 1

In particular, finite sets are countable and for these sets it is convenient to interpret $\mathrm{card}\,(X)$ as the number of elements in X:

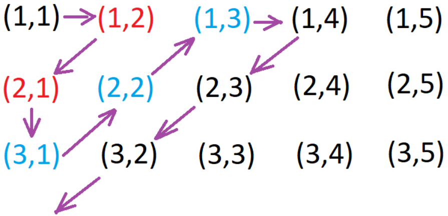
$$\mathrm{card}\,(X) = n \iff \mathrm{card}\,(X) = \mathrm{card}\,(\{1, 2, \dots, n\}).$$

Example 2

If X is countable but not finite, we say that X is **countably infinite**.

Countable sets - example

The set $\mathbb{N} \times \mathbb{N}$ is countable, i.e. $\mathrm{card}\,(\mathbb{N} \times \mathbb{N}) = \mathrm{card}\,(\mathbb{N})$. Note that



can be listed as a sequence

$(1,1), (1,2), (2,1), (1,3), (2,2), (3,1), (1,4), (2,3), (3,2), (4,1), (5,1), (4,2), \ldots$

establishing a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

Proposition

Proposition

- (a) If X and Y are countable, so is $X \times Y$.
- (b) If A is countable and X_α is countable for all $\alpha \in A$, then

$$\bigcup_{\alpha \in A} X_\alpha \text{ is countable.}$$

- (c) If X is countably infinite then $\operatorname{card}(X) = \operatorname{card}(\mathbb{N})$.

Proof of (a). To prove (a) it suffices to show that $\operatorname{card}(\mathbb{N} \times \mathbb{N}) = \operatorname{card}(\mathbb{N})$, but it was shown in the previous example.

Proof of (b). For each $\alpha \in A$ there is a surjection $f_\alpha : \mathbb{N} \to \mathbb{X}_\alpha$ (here we have used the axiom of choice).

Proof of Proposition 1/2

Then the map $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{\alpha \in A} X_\alpha$ defined by

$$f(n, \alpha) = f_\alpha(n)$$

is surjective and we are done. Alternatively, to prove

$$\mathrm{card}\left(\bigcup_{\alpha \in A} X_\alpha\right) \leq \mathrm{card}\,(\mathbb{N}),$$

we can also proceed as follows

$$X_1 = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, \ldots\}$$

$$X_2 = \{x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, \ldots\}$$

$$X_3 = \{x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4}, \ldots\}$$

$$\ldots$$

$$\bigcup_{j \in \mathbb{N}} X_j = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{1,3}, x_{2,2}, x_{3,1}, \ldots\}.$$

Proof of Proposition 2/2

Proof of (c). We can assume that X is an infinite subset of $\mathbb{N}$.

Let $f(1)$ be the smallest element of X and define inductively

$$f(n) = \min \big\{ X \setminus \{f(1), f(2), \ldots, f(n-1)\} \big\}.$$

Then it can be easily verified that f is a bijection from $\mathbb{N}$ to X. $\square$

Corollaries

Corollary

$\mathbb{Z}$ and $\mathbb{Q}$ are countable.

Proof.

$$\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n \ : \ n \in \mathbb{N}\},$$

- Note that
$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} \{\frac{m}{n} \ : \ m \in \mathbb{Z}\}.$$
- Note also that
$$\mathrm{card}\,(\mathbb{Q}) = \mathrm{card}\,(\mathbb{N}) = \mathrm{card}\,(\mathbb{N} \cup \{0\}) = \mathrm{card}\,(\mathbb{Z}).$$

Countable sets - example

Example

Prove that the following set is countable

$$X = \{(n, m, k) \ : \ n, m, k \in \mathbb{N}\}.$$

Solution. Fix $k \in \mathbb{N}$. It is clear that

$$f(n, m) = (n, m, k)$$

is a bijection between $\mathbb{N} \times \mathbb{N}$ and $A_k = \{(n, m, k) \ : \ n, m \in \mathbb{N}\}$, so A_k is countable. Then we write

$$X = \bigcup_{k \in \mathbb{N}} A_k$$

and use the previous theorem. $\square$

Countable sets - example

Example

Prove that the set $\mathbb{Q} \times \mathbb{Q}$ is countable.

Solution. Fix $q \in \mathbb{Q}$. It is easy to check that $f(x) = (x, q)$ is bijection between $\mathbb{Q}$ and $A_q = \{(r, q) \ : \ r \in \mathbb{Q}\}$. Hence A_q is countable. We write

$$\mathbb{Q} \times \mathbb{Q} = \bigcup_{q \in \mathbb{Q}} A_q$$

and use the previous theorem. □

Proposition

Proposition

$\{0,1\}^{\mathbb{N}_0}$ is uncountable.

Proof. Suppose that the set $\{0,1\}^{\mathbb{N}_0}$ is countable, then

$$\{0,1\}^{\mathbb{N}_0} = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}.$$

$$\begin{aligned}
\alpha_0 &: \quad \alpha_0(0), \alpha_0(1), \alpha_0(2), \ldots \\
\alpha_1 &: \quad \alpha_1(0), \alpha_1(1), \alpha_1(2), \ldots \\
\alpha_2 &: \quad \alpha_2(0), \alpha_2(1), \alpha_2(2), \ldots
\end{aligned}$$

$$\ldots$$

consider a new sequence $\Delta : \mathbb{N}_0 \to \{0,1\}$ defined by $\Delta(n) = 1 - \alpha_n(n)$. Then $\Delta \neq \alpha_n$ for all $n \in \mathbb{N}_0$, thus $\Delta \notin \{\alpha_n \ : \ n \in \mathbb{N}_0\}$, but $\Delta \in \{0,1\}^{\mathbb{N}_0}$, contradiction. □

Cardinality continuum

Cardinality continuum

A set X is said to have cardinality **continuum** if

$$\operatorname{card}(X) = \operatorname{card}(\mathbb{R}).$$

We shall write $\operatorname{card}(X) = \mathfrak{c}$ iff $\operatorname{card}(X) = \operatorname{card}(\mathbb{R})$.

Theorem

$$\operatorname{card}(P(\mathbb{N})) = \operatorname{card}(\{0,1\}^{\mathbb{N}}) = \operatorname{card}(\mathbb{R}).$$

Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}})$: 1/2

We first show that

$$\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}).$$

Define $F : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ by setting

$$F(A) = \chi_A \in \{0,1\}^{\mathbb{N}},$$

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

Example

If $A = \{1,2,3,5,7,9\}$, then χ_A can be identified with the sequence

$$(\underbrace{1}_{1}, \underbrace{1}_{2}, \underbrace{1}_{3}, \underbrace{0}_{4}, \underbrace{1}_{5}, \underbrace{0}_{6}, \underbrace{1}_{7}, \underbrace{0}_{8}, \underbrace{1}_{9}, \underbrace{0}_{10}, \underbrace{0}_{11}, \ldots)$$

Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}})$: 2/2

Our aim is to show that F is a bijection.

- Let $A, B \subseteq \mathbb{N}$ be such that $A \neq B$, then we show that $F(A) \neq F(B)$. Since $A \neq B$ (wlog) there exists $x_0 \in A \setminus B$, thus

$$1 = \chi_A(x_0) \neq \chi_B(x_0) = 0 \quad \iff \quad F(A) \neq F(B),$$

  which proves that $F : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ is injective.

- To show that F is surjective take $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ and consider

$$E = \{n \in \mathbb{N} \,:\, \alpha_n = 1\}.$$

  then

$$F(E) = \chi_E = \alpha \quad \iff \quad 1 = \alpha_n = \chi_E(n) = 1.$$

Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R})$: 1/4

It remains to prove that

$$\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R}).$$

We first show that $\mathrm{card}\,(P(\mathbb{N})) \leq \mathrm{card}\,(\mathbb{R})$.

Definition

We will use the convention that

$$\sum_{n\in A} a_n = \sup\left\{\sum_{n\in F} a_n \;:\; F \subseteq A \text{ and } F \text{ is finite}\right\}.$$

We now construct an injection $f : P(\mathbb{N}_0) \to \mathbb{R}$ by setting

$$f(A) = \sum_{n\in A} \frac{2}{3^{n+1}} \quad \text{for any} \quad A \subseteq \mathbb{N}_0.$$

Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R})$: 2/4

First we have to show that f is well defined, i.e. $f(A)$ is finite for every $A \subseteq \mathbb{N}_0$. Since $f(A) \leq f(\mathbb{N}_0)$ for all $A \subseteq \mathbb{N}_0$ it suffices to show that

$f(\mathbb{N}_0)$ is finite.

Let $[N] = \{0, 1, 2, \ldots, N\}$ and observe that $f(\mathbb{N}_0) = \sup_{N \in \mathbb{N}_0} f([N])$. Moreover, one has

$$f([N]) = \sum_{n \in [N]} \frac{2}{3^{n+1}} = \frac{2}{3} \frac{1 - \frac{1}{3^{N+1}}}{1 - \frac{1}{3}} = 1 - \frac{1}{3^{N+1}} \leq 1.$$

Geometric sequence

Here we have used the formula $1 + x + x^2 + \ldots + x^N = \frac{1-x^{N+1}}{1-x}$, which holds for all $|x| < 1$.

Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R})$: 3/4

In fact, one can prove that $f(\mathbb{N}_0) = 1$ (check this!). Now we show that $f : P(\mathbb{N}_0) \to \mathbb{R}$ is injective. Let $A, B \subseteq \mathbb{N}$ be such that $A \neq B$. Let

$$n_0 = \min\{n \in \mathbb{N}_0 \colon \chi_A(n) \neq \chi_B(n)\}.$$

Wlog we can assume $n_0 \in B \setminus A$. Then

$$\begin{aligned} f(A) = \sum_{n \in A} \frac{2}{3^{n+1}} &= \sum_{\substack{n \in A \\ n < n_0}} \frac{2}{3^{n+1}} + \sum_{\substack{n \in A \\ n > n_0}} \frac{2}{3^{n+1}} \\ &\leq \sum_{\substack{n \in B \\ n < n_0}} \frac{2}{3^{n+1}} + \sum_{n \in (n_0, \infty)} \frac{2}{3^{n+1}} \\ &< \sum_{\substack{n \in B \\ n < n_0}} \frac{2}{3^{n+1}} + \frac{2}{3^{n_0+1}} \leq f(B). \end{aligned}$$

Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R})$: 4/4

- We have used $f((n_0, \infty) \cap \mathbb{N}_0) = \frac{1}{3^{n_0+1}} < \frac{2}{3^{n_0+1}}$ (check this!). Hence $f(A) \neq f(B)$ as desired, yielding $\mathrm{card}\,(P(\mathbb{N}_0)) \leq \mathrm{card}\,(\mathbb{R})$.
- Since $\mathrm{card}\,(\mathbb{N}_0) = \mathrm{card}\,(\mathbb{Q})$ thus $\mathrm{card}\,(P(\mathbb{N}_0)) = \mathrm{card}\,(P(\mathbb{Q}))$. Now define $g : \mathbb{R} \to P(\mathbb{Q})$ by

$$g(x) = \{r \in \mathbb{Q} \colon r < x\} \quad \text{for any} \quad x \in \mathbb{R}.$$

It is easy to see that g is injective, thus

$$\mathrm{card}\,(\mathbb{R}) \leq \mathrm{card}\,(P(\mathbb{Q})) = \mathrm{card}\,(P(\mathbb{N}_0)).$$

By the **Cantor–Bernstein–Schröder** theorem

$$\mathrm{card}\,(\mathbb{R}) = \mathrm{card}\,(P(\mathbb{N}_0)) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}). \quad \square$$

Remarks

A. If $\mathrm{card}\,(X) \geq \mathfrak{c}$, then X is uncountable.

B. $\mathrm{card}\,(\mathbb{R} \times \mathbb{R}) = \mathrm{card}\,(\mathbb{R})$.

C. $\mathrm{card}\,(\{0,1\}^{\mathbb{N}_0} \times \{0,1\}^{\mathbb{N}_0}) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}_0})$.

D. $\mathrm{card}\,(\mathbb{R}^k) = \mathrm{card}\,(\mathbb{R})$ for any $k \in \mathbb{N}$.

E. If $\mathrm{card}\,(X) \leq \mathfrak{c}$ and $\mathrm{card}\,(Y) \leq \mathfrak{c}$, then $\mathrm{card}\,(X \times Y) \leq \mathfrak{c}$.

F. If $\mathrm{card}\,(A) \leq \mathfrak{c}$ and $\mathrm{card}\,(X_\alpha) \leq \mathfrak{c}$ for any $\alpha \in A$, then

$$\mathrm{card}\left(\bigcup_{\alpha \in A} X_\alpha\right) \leq \mathfrak{c}.$$

Uncountable sets - example

Example

Prove that $\text{card}\,(\mathbb{R}) = \text{card}\,([0,1])$.

Solution. Define $f : \mathbb{R} \to [0,1]$ by

$$f(x) = \frac{x}{|x|+1}.$$

One can verify that f is a bijection between $\mathbb{R}$ and $[0,1]$. $\square$

Uncountable sets - example

Example

Determine $\mathrm{card}\,(X)$, where X is

$$\{[n, n+1)\ :\ n \in \mathbb{N}\}.$$

Solution. We will prove that $\mathrm{card}\,(X) = \mathrm{card}\,(\mathbb{N})$. Let us define the function

$$f([n, n+1)) = n.$$

It is easy to verify that f is bijection between $\mathbb{N}$ and X. □

Uncountable sets - example

Example

Determine $\mathrm{card}\,(X)$, where X is any infinite set of pairwise disjoint closed intervals.

Solution. We will prove that X is **countably infinite**. Let us define the function $f : X \to \mathbb{Q}$ by setting

$$f(I) = q,$$

where q is any rational number contained in interval $I \in X$. Then f is injective, so $\mathrm{card}\,(X) \leq \mathrm{card}\,(\mathbb{N})$. □