Lesson 9

Inequality between the arithmetic and geometric means and other useful inequalities

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Geometric and arithmetic means

Let $a_1, a_2, \ldots, a_n \geq 0$ be given.

Arithmetic mean

We define **arithmetic mean** of $a_1, a_2, \ldots, a_n$ by

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

Geometric mean

We define **geometric mean** of $a_1, a_2, \ldots, a_n$ by

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}.$$

Geometric Mean vs Arithmetic Mean

Theorem

For any $n \in \mathbb{N}$ we have

$$G_n \leq A_n.$$

Proof. For $n = 2$ observe that

$$(a - b)^2 \geq 0,$$

since

$$a^2 - 2ab + b^2 \geq 0 \quad \Longleftrightarrow \quad ab \leq \frac{a^2 + b^2}{2}.$$

Taking $a = \sqrt{a_1}$ and $b = \sqrt{a_2}$ we obtain

$$A_2 = \frac{a_1 + a_2}{2} = \frac{(\sqrt{a_1})^2 + (\sqrt{a_2})^2}{2} \geq \sqrt{a_1 a_2} = G_2.$$

Cases $n = 4$ and $n = 8$ suggest induction

Case $n = 4$**.** Note that

$$A_4 = \frac{a_1 + a_2 + a_3 + a_4}{4} = \frac{1}{2}\left(\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2}\right)$$

$$\underbrace{\geq}_{A_2 \geq G_2} \left((a_1 a_2)^{1/2} (a_3 a_4)^{1/2}\right)^{1/2} = (a_1 a_2 a_3 a_4)^{1/4} = G_4.$$

Case $n = 8$**.** Let us use $A_4 \geq G_4$ and $A_2 \geq G_2$ to prove $A_8 \geq G_8$.

$$A_8 = \frac{a_1 + \ldots + a_8}{8} = \frac{1}{2}\left(\frac{a_1 + \ldots + a_4}{4} + \frac{a_5 + \ldots + a_8}{4}\right)$$

$$\underbrace{\geq}_{A_2 \geq G_2} \left(\frac{a_1 + \ldots + a_4}{4} \frac{a_5 + \ldots + a_8}{4}\right)^{1/2}$$

$$\underbrace{\geq}_{A_4 \geq G_4} \left((a_1 \ldots a_4)^{1/4} (a_5 \ldots a_8)^{1/4}\right)^{1/2} = (a_1 \ldots a_8)^{1/8} = G_8.$$

Claim and base step

We first use induction to prove

$$A_{2^n} \geq G_{2^n}$$

for all $n \in \mathbb{N}$.

Base step. For $n = 2$ the inequality is true as

$$A_2 = \frac{a_1 + a_2}{2} \geq (a_1 a_2)^{1/2} = G_2.$$

Inductive step

Let $P(n)$ be the statement that $A_{2^n} \geq G_{2^n}$ holds for some $n \in \mathbb{N}$.

Inductive step. Now we prove that $P(n) \implies P(n+1)$. Indeed,

$$
\begin{aligned}
A_{2^{n+1}} =& \frac{a_1 + \ldots + a_{2^{n+1}}}{2^{n+1}} = \frac{1}{2}\left(\frac{a_1 + \ldots + a_{2^n}}{2^n} + \frac{a_{2^n+1} + \ldots + a_{2^{n+1}}}{2^n}\right) \\
&\underbrace{\geq}_{A_2 \geq G_2} \left(\frac{a_1 + \ldots + a_{2^n}}{2^n} \frac{a_{2^n+1} + \ldots + a_{2^{n+1}}}{2^n}\right)^{1/2} \\
&\underbrace{\geq}_{A_{2^n} \geq G_{2^n}} \left((a_1 \ldots a_{2^n})^{1/2^n} (a_{2^n+1} \ldots a_{2^{n+1}})^{1/2^n}\right)^{1/2} \\
&= (a_1 \ldots a_{2^{n+1}})^{1/2^{n+1}} = G_{2^{n+1}}.
\end{aligned}
$$

Proof of GM vs AM inequality

Now we have to show that

$$A_n \geq G_n$$

for all $n \in \mathbb{N}$.

We first observe that the following downwards induction holds.

Let $Q(n)$ be the statement that

$$A_n \geq G_n$$

holds for some $n \in \mathbb{N}$. Then

$$Q(n-1)$$

is also true.

This will follow from the so-called **bootstrap phenomenon**.

Bootstrap phenomenon

Note that (by $A_n \geq G_n$ with $a_1, a_2, \ldots, a_{n-1}, a_n = A_{n-1}$) one has

$$\frac{a_1 + \ldots + a_{n-1} + A_{n-1}}{n} \geq (a_1 \cdot \ldots \cdot a_{n-1} \cdot A_{n-1})^{1/n}.$$

But

$$\frac{a_1 + \ldots + a_{n-1} + A_{n-1}}{n} = \frac{(n-1)A_{n-1} + A_{n-1}}{n} = A_{n-1}.$$

Thus we have shown

Bootstrapping inequality

$$A_{n-1} \geq (a_1 \cdot \ldots \cdot a_{n-1})^{1/n} A_{n-1}^{1/n}.$$

Hence

$$A_{n-1}^{1-1/n} \geq (a_1 \cdot \ldots \cdot a_{n-1})^{1/n} = G_{n-1}^{(n-1)/n},$$

thus $A_{n-1} \geq G_{n-1}$, which means that $Q(n-1)$ holds.

Geometric Mean vs Arithmetic Mean: 1/2

Now we can show that

Claim (⋆)

$$A_n \geq G_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. We know:

1. $A^{2^m} \geq G_{2^m}$ for all $m \in \mathbb{N}$,
2. if $A_k \geq G_k$ holds for some $k \in \mathbb{N}$, then also holds for $k-1$, i.e. $A_{k-1} \geq G_{k-1}$ is true.

Concluding, we can easily prove Claim (⋆). Fix $n \in \mathbb{N}$ and choose the smallest $m \in \mathbb{N}$ so that

$$2^{m-1} < n \leq 2^m.$$

By (1) we know $A_{2^m} \geq G_{2^m}$ holds.

Geometric Mean vs Arithmetic Mean: 2/2

Claim (⋆)

$$A_n \geq G_n \quad \text{for all} \quad n \in \mathbb{N}.$$

By (2) with $k = 2^m$ we deduce

$$A_{2^m} \geq G_{2^m} \quad \text{implies} \quad A_{2^m-1} \geq G_{2^m-1}.$$

Repeating

$$A_{2^m-1} \geq G_{2^m-1} \quad \text{implies} \quad A_{2^m-2} \geq G_{2^m-2}.$$

We now apply (2) as many times until we reach $A_n \geq G_n$ and the proof is finally completed. □

Means

Let $a_1, a_2, \ldots, a_n > 0$ be given. We have the following means.

Arithmetic mean

$$A_n = \frac{a_1 + \ldots + a_n}{n};$$

Geometric means

$$G_n = (a_1 \cdot \ldots \cdot a_n)^{1/n};$$

Harmonic mean

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}};$$

Means

Quadratic mean

$$Q_n = \left(\frac{a_1^2 + \ldots + a_n^2}{n}\right)^{1/2}.$$

Example

For $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ we have

$$A_3 = \frac{1+2+4}{3} = \frac{7}{3} \approx 2,333\ldots, \quad G_3 = \sqrt[3]{1\cdot 2\cdot 4} = 2,$$

$$H_3 = \frac{3}{1+\frac{1}{2}+\frac{1}{4}} = \frac{12}{7} \approx 1,714\ldots, \quad Q_3 = \sqrt{7} \approx 2,645\ldots$$

$$\min(1,2,4) = 1, \qquad \max(1,2,4) = 4.$$

Theorem

Theorem

For all $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n > 0$ we have

$$\min(a_1, \ldots, a_n) \leq H_n \leq G_n \leq A_n \leq Q_n \leq \max(a_1, \ldots, a_n).$$

Proof. We will proceed in several steps.

- We have proved that $A_n \geq G_n$.
- To prove $H_n \leq G_n$ we apply $A_n \geq G_n$ with

$$\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}.$$

We obtain

$$G_n^{-1} = \left(\frac{1}{a_1} \cdot \frac{1}{a_2} \cdot \ldots \cdot \frac{1}{a_n}\right)^{1/n} \leq \frac{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}{n} = H_n^{-1},$$

thus $H_n \leq G_n$.

Proof: 1/2

- To prove inequality $A_n \le Q_n$ consider the relation

$$\begin{aligned}(a_1 + a_2 + \ldots + a_n)^2 = {} & a_1^2 + a_2^2 + \ldots + a_n^2 \\ & + 2(a_1 a_1 + a_1 a_3 + \ldots + a_1 a_n) \\ & + 2(a_2 a_3 + a_2 a_4 + \ldots + a_2 a_n) \\ & + \ldots + 2(a_{n-2} a_{n-1} + a_{n-2} a_n) + 2a_{n-1} a_n.\end{aligned}$$

Since $2a_i a_j \le a_i^2 + a_j^2$, thus

$$(a_1 + \ldots + a_n)^2 \le n(a_1^2 + \ldots + a_n^2).$$

Hence

$$a_1 + \ldots + a_n \le \left(n(a_1^2 + \ldots + a_n^2)\right)^{1/2},$$

and consequently $A_n \le Q_n$.

Proof: 2/2

- Finally wlog suppose that

$$0 < a_1 \le a_2 \le \ldots \le a_n.$$

then $a_1 = \min(a_1, \ldots, a_n)$, and $a_n = \max(a_1, \ldots, a_n)$. Hence

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \ge n\frac{a_1}{n} = a_1,$$

and

$$Q_n = \left(\frac{a_1^2 + \ldots + a_n^2}{n}\right)^{1/2} \le \left(\frac{na_n^2}{n}\right)^{1/2} = a_n.$$

The proof is completed. □

AM-GM inequality - example

Example

Prove that for any $x, y, z > 0$ we have

$$\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy} \geq 3.$$

Solution. Consider the numbers $\frac{x^2}{yz}, \frac{y^2}{xz}, \frac{z^2}{xy}$. Then

$$A_3 = \frac{\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy}}{3},$$

$$G_3 = \sqrt[3]{\frac{x^2}{yz} \cdot \frac{y^2}{xz} \cdot \frac{z^2}{xy}} = 1,$$

so our inequality is a consequence of $A_3 \geq G_3$. □

AM-GM inequality - example

Example

If the product of n positive real numbers is 1, then their sum is at least n.

Solution. Let $a_1, \ldots, a_n > 0$ be the numbers such that

$$G_n = \sqrt[n]{a_1 \cdots a_n} = 1,$$

so by $A_n \geq G_n$,

$$a_1 + \ldots + a_n \geq nG_n = n.$$

□

Bernoulli inequality: 1/2

Bernoulli inequality

If $x > -1$ and $n \in \mathbb{N}$, then one has

$$(1+x)^n \geq 1 + nx.$$

Proof. We will use $A_n \geq G_n$ with

$$a_1 = a_2 = \ldots = a_{n-1} = 1 \quad \text{and} \quad a_n = 1 + nx.$$

Indeed,

$$A_n = \frac{a_1 + \ldots + a_n}{n} = \frac{\overbrace{1 + \ldots + 1}^{n-1 \text{ times}} + 1 + nx}{n} = \frac{n(1+x)}{n} = 1 + x.$$

Bernoulli inequality: 2/2

On the other hand

$$(1+x) = A_n \geq G_n = \Bigg(\overbrace{1 \cdot 1 \cdot \ldots \cdot 1}^{n-1 \text{ times}} \cdot (1+nx)\Bigg)^{1/n} = (1+nx)^{1/n},$$

which implies

$$(1+x)^n \geq 1+nx,$$

and we are done. □

Bernoulli inequality: generalization

Our aim will be to built tools and generalize Bernoulli's inequality. We show that the following is true.

Bernoulli inequality - generalization

If $-1 < x \neq 0$ and $a > 1$ or $a < 0$, then

$$(1+x)^a > 1 + ax.$$

If $-1 < x \neq 0$ and $0 < a < 1$, then

$$(1+x)^a < 1 + ax.$$

Cauchy–Schwarz inequality

Cauchy–Schwarz inequality

For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ one has

$$\left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sum_{j=1}^{n} a_j^2 \right) \left(\sum_{j=1}^{n} b_j^2 \right).$$

Proof. Consider the polynomial

$$0 \leq (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \ldots + (a_n x + b_n)^2 =$$
$$(a_1^2 + \ldots + a_n^2)x^2 + 2(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)x + (b_1^2 + \ldots + b_n^2).$$

Since the polynomial is nonnegative

$$\Delta = 4(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) - 4(a_1 b_1 + \ldots + a_n b_n) \geq 0$$

and we are done. □

Triangle's inequality

Triangle's inequality

For all $x, y \in \mathbb{R}$ one has

$$|x + y| \leq |x| + |y|.$$

Consequently, one has

$$\big||x| - |y|\big| \leq |x - y| \leq |x| + |y|.$$

Proof is an easy exercise.

Minkowski's inequality: 1/3

Minkowski's inequality

For any real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$ one has

$$\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}.$$

Proof. Let

$$S_n = \left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2}.$$

Minkowski's inequality: 2/3

Then

$$S_n^2 = \sum_{j=1}^{n} |a_j + b_j|^2 = \sum_{j=1}^{n} |a_j + b_j||a_j + b_j|$$
$$\leq \sum_{j=1}^{n} |a_j + b_j||a_j| + \sum_{j=1}^{n} |a_j + b_j||b_j|.$$

By the Cauchy–Schwarz inequality,

$$S_n^2 \leq$$
$$\underbrace{\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2}}_{=S_n} \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \underbrace{\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2}}_{=S_n} \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}$$

Minkowski's inequality: 3/3

$$= S_n \left(\left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \right).$$

Thus we have proved **a bootstrap inequality**, i.e.

$$S_n^2 \leq S_n \left(\left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \right).$$

Hence

$$S_n \leq \left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2}.$$

□

Hölder's inequality

Hölder's inequality

Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ one has

$$\sum_{j=1}^{n} |a_j b_j| \leq \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q \right)^{1/q}.$$

This inequality is an extension of the Cauchy–Schwarz inequality. However at the moment we do not have tools to prove this inequality.