Lecture 10

Absolute and conditional convergence of infinite series

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Absolute convergence

Absolute convergence

The series $\sum_{n=1}^{\infty} a_n$ is said **to converge absolutely** if the series

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

converges.

Theorem

If $\sum_{n=1}^{\infty} |a_n| < \infty$, then $|\sum_{n=1}^{\infty} a_n| < \infty$.

Proof. The claim follows from the Cauchy Criterion, since

$$\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k|$$

and we are done. □

Conditional convergence

Conditional convergence

If the series $\sum_{n=1}^{\infty} a_n$ converges but

$$\sum_{n=1}^{\infty} |a_n| = \infty$$

diverges then we say that $\sum_{n=1}^{\infty} a_n$ **converges conditionally**.

Example 1

For series with positive terms, absolute convergence is the same as convergence.

Example 2

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

Anharmonic series

Anharmonic series

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges conditionally.

It is easy to see that

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

To prove $\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right| < \infty$ we will show a more general result.

Summation by parts (Abel summation formula)

Abel summation formula

Given two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ set

$$A_n = \sum_{k=0}^{n} a_k \quad \text{for} \quad n \geq 0, \quad \text{and} \quad A_{-1} = 0.$$

Then if $0 \leq p \leq q$ one has

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n \left(b_n - b_{n+1}\right) + A_q b_q - A_{p-1} b_p.$$

Proof of Abel summation formula

Proof: Note that

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} \underbrace{(A_n - A_{n-1})}_{a_n} b_n = \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n$$

$$= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1}$$

$$= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

The proof follows. □

Theorem

Dirichlet's test

Suppose that

- (a) The partial sums $A_n = \sum_{k=1}^{n} a_k$ of $(a_n)_{n\in\mathbb{N}}$ form a bounded sequence.
- (b) $b_0 \geq b_1 \geq b_2 \geq b_3 \geq \ldots$,
- (c) $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof. Choose $M \geq 0$ so that $|A_n| \leq M$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ so that

$$b_{N_\varepsilon} < \frac{\varepsilon}{2M},$$

since $\lim_{n\to\infty} b_n = 0$.

Proof

For $N_\varepsilon \leq p \leq q$, by the summation by parts formula, one has

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n\left(b_n - b_{n+1}\right) + A_q b_q - A_{p-1} b_p\right| \\ &\leq \left|\sum_{n=p}^{q-1} A_n\left(b_n - b_{n+1}\right)\right| + |A_q b_q| + |A_{p-1} b_p| \\ &\leq M\sum_{n=p}^{q-1} |(b_n - b_{n+1})| + M b_q + M b_p \leq 2Mb_p \leq 2Mb_{N_\varepsilon} < \varepsilon.\end{aligned}$$

since

$$b_p - b_q = \sum_{n=p}^{q-1} |(b_n - b_{n+1})| = \sum_{n=p}^{q-1} (b_n - b_{n+1})$$

$$= (b_p - b_{p+1}) + (b_{p+1} - b_{p+2}) + (b_{p+2} - b_{p+3}) + \ldots + b_{q-1} - b_q. \quad \square$$

Anharmonic series

We now show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. Let

$$a_n = (-1)^n, \quad \text{and} \quad b_n = \frac{1}{n}$$

in the previous theorem. We see that

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right| = \left| \sum_{n=1}^{\infty} a_n b_n \right| < \infty$$

since

$$|A_n| = \left| \sum_{k=1}^{n} (-1)^k \right| \leq 1.$$

Alternating Series Test

A more general result can be proved:

Alternating Series Test

Let $(a_n)_{n\in\mathbb{N}}$ be such that

(i) $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots,$

(ii) $\lim_{n\to\infty} a_n = 0.$

Then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.

Proof. We apply the previous theorem.

Example

Exercise

Determine if the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ converges and converges absolutely.

Solution. Let $a_n = \frac{1}{\sqrt{n^2+1}}$.

- We have
$$a_n \geq \frac{1}{\sqrt{4n^2}} = \frac{1}{2n},$$
so the series **does not converges absolutely**, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.
- On the other hand, we have
$$a_n \geq a_{n+1} \quad \text{and} \quad \lim_{n\to\infty} a_n = 0,$$
so the assumptions of the previous theorem are satisfied. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$ **converges conditionally**. □

Product of two series

Definition

Given $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ we set

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0$$

and call $\sum_{n=0}^{\infty} c_n$ **the product of the two given series**

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n.$$

Theorem

Theorem

Suppose that

1. (a) $\sum_{n=0}^{\infty} |a_n| < \infty$,
2. (b) $\sum_{n=0}^{\infty} a_n = A$,
3. (c) $\sum_{n=0}^{\infty} b_n = B$,
4. (d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

Proof: 1/2

Proof. Let

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k,$$

$$\beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \ldots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \ldots + a_n(B + \beta_0). \end{aligned}$$

Thus

$$C_n = A_n B + \underbrace{a_0\beta_n + a_1\beta_{n-1} + \ldots + a_n\beta_0}_{\gamma_n}.$$

We will show that $C_n \xrightarrow[n\to\infty]{} AB$. Since $A_n B \xrightarrow[n\to\infty]{} AB$ it suffices to prove that $\gamma_n \xrightarrow[n\to\infty]{} 0$.

Proof: 2/2

Set $\alpha = \sum_{n=0}^{\infty} |a_n|$. Let $\varepsilon > 0$ be given. By (b) $\beta_n \xrightarrow[n\to\infty]{} 0$, thus we find $N \in \mathbb{N}$ such that

$$|\beta_n| < \varepsilon \quad \text{for} \quad n \geq N,$$

then

$$\begin{aligned}|\gamma_n| &\leq |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N+1} + \ldots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \ldots + \beta_N a_{n-N}| + \varepsilon\alpha.\end{aligned}$$

We keep $N \in \mathbb{N}$ fixed and letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty} |\gamma_n| \leq \varepsilon\alpha$$

since $a_k \xrightarrow[k\to\infty]{} 0$. But $\varepsilon > 0$ is arbitrary we get

$$\liminf_{n\to\infty} |\gamma_n| = \limsup_{n\to\infty} |\gamma_n| = 0 = \lim_{n\to\infty} |\gamma_n|. \qquad \square$$

Remark

Remark

If $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, $\sum_{n=0}^{\infty} c_n = C$, and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \ldots + a_{n-1} b_1 + a_n b_0$$

then $C = AB$.

Rearrangements

Rearrangement

Let $(k_n)_{n\in\mathbb{N}}$ be a sequence in which every positive integer appears once and only once. Setting

$$a_n' = a_{k_n}$$

we say that $\sum_{n=1}^{\infty} a_n'$ is **rearrangement** of $\sum_{n=1}^{\infty} a_n$.

Example

- Consider the convergent series

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} \underbrace{-\frac{1}{4} + \frac{1}{5}}_{<0} \underbrace{-\frac{1}{6} + \frac{1}{7}}_{<0} - \dots$$

Example

- Consider also a rearrangement S' of S given by:

$$S' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots + \\ = \sum_{k=1}^{\infty} \left(\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} \right)$$

- Observe that $S < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0 \quad \text{for all} \quad k \in \mathbb{N}.$$

- If S'_n is the partial sum of S' then

$$S'_3 < S'_6 < S'_9 < \ldots$$

  hence $\limsup_{n\to\infty} S'_n > S'_3 = \frac{5}{6}$.
- Thus S' **does not converge** to $S < \frac{5}{6}$.

Theorem

Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series that converges conditionally. Suppose that

$$-\infty \leq \alpha \leq \beta \leq +\infty.$$

Then there exists a reaarangement $\sum_{n=0}^{\infty} a'_n$ with partial sums s'_n so that

$$\liminf_{n\to\infty} s'_n = \alpha, \quad \text{and} \quad \limsup_{n\to\infty} s'_n = \beta.$$

Proof. Let

$$p_n = \frac{|a_n| + a_n}{2} \geq 0, \quad \text{and} \quad q_n = \frac{|a_n| - a_n}{2} \geq 0.$$

Then

$$p_n - q_n = a_n \quad \text{and} \quad p_n + q_n = |a_n|.$$

Proof: 1/5

Claim

The series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ both diverge.

- Indeed, if both were convergent then

$$+\infty = \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} (p_n + q_n) < +\infty,$$

contradiction.

- Since

$$\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} (p_n - q_n) = \sum_{n=1}^{N} p_n - \sum_{n=1}^{N} q_n,$$

then divergence of $\sum_{n=1}^{\infty} p_n$ and convergence of $\sum_{n=1}^{\infty} q_n$ (or vice versa) implies divergence of $\sum_{n=1} a_n$, contradiction.

Proof: 2/5

- Let $P_1, P_2, \ldots$ denote the nonnegative terms of $\sum_{n=1}^{\infty} a_n$ in the order in which they occur.
- Let $Q_1, Q_2, \ldots$ be the absolute values of the negative terms of $\sum_{n=1}^{\infty} a_n$ also in their original order.
- The series $\sum_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} Q_n$ differ from $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ also by zero terms and therefore thay also diverge.

Claim

We shall construct $(m_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ such that the series

$$P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} \\ + P_{m+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots - Q_{k_2} + \ldots$$

which is a rearrangement of $\sum_{n=1}^{\infty} a_n$ satisfies

$$\liminf_{n\to\infty} s_n' = \alpha \quad \text{and} \quad \limsup_{n\to\infty} s_n' = \beta.$$

Proof: 3/5

- Choose $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ so that $\alpha_n < \beta_n$ with $\beta_1 > 0$ and

$$\alpha_n \xrightarrow[n\to\infty]{} \alpha, \quad \text{and} \quad \beta_n \xrightarrow[n\to\infty]{} \beta.$$

- Let m_1, k_1 be the smallest integers such that

$$P_1 + \ldots + P_{m_1} > \beta_1 \quad \text{and} \quad P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} < \alpha_1.$$

- Let m_2, k_2 be the smallest integers such that

$$P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} + P_{m_1+1} + \ldots + P_{m_2} > \beta_2,$$

and

$$P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} + P_{m+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots - Q_{k_2} < \alpha_2.$$

and we continue this way.

Proof: 4/5

- This is possible since

$$\left|\sum_{n=1}^{\infty} P_n\right| = \infty \quad \text{and} \quad \left|\sum_{n=1}^{\infty} Q_n\right| = \infty.$$

- If x_n, y_n are the partial sums of

$$P_1 + \ldots + P_{m_1} - Q_1 - \ldots - Q_{k_1} \\ + P_{m+1} + \ldots + P_{m_2} - Q_{k_1+1} - \ldots - Q_{k_2}$$

whose last terms respectively are P_{m_n} and Q_{k_n} then

$$|x_n - \beta_n| \le P_{m_n} \quad \text{and} \quad |y_n - \alpha_n| \le Q_{k_n}.$$

Proof: 5/5

- Since
$$\beta_n < x_n \leq (x_n - P_{m_n}) + P_{m_n} \leq \beta_n + P_{m_n},$$
then
$$0 < x_n - \beta_n \leq P_{m_n}.$$
- Since $P_n \xrightarrow[n\to\infty]{} 0$ and $Q_n \xrightarrow[n\to\infty]{} 0$ we see that
$$x_n \xrightarrow[n\to\infty]{} \beta.$$
- Similarly we conclude that
$$y_n \xrightarrow[n\to\infty]{} \alpha.$$
- Finally it is clear that no number less that α or greater that β can be subsequential limit of the partial sums of s_n'.

This completes the proof of the theorem. □

Theorem

If $\sum_{n=1}^{\infty} |a_n| < \infty$, then every rearrangement of $\sum_{n=1}^{\infty} a_n$ converge to the same limit.

Proof. Let $\sum_{n=1}^{\infty} a'_n$ be a rearrangement of $\sum_{n=1}^{\infty} a_n$ with partial sums s'_n. Given $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that $m \geq n \geq N_\varepsilon$ implies

$$\sum_{k=n}^{m} |a_k| < \varepsilon.$$

Now choose $p \in \mathbb{N}$ such that

$$\{1, 2, \ldots, N_\varepsilon\} \subseteq \{k_1, k_2, \ldots, k_p\}; \qquad \text{here} \qquad a'_n = a_{k_n}.$$

If $n > p$ then the numbers $a_1, \ldots, a_{N_\varepsilon}$ will cancel in the difference $s_n - s'_n$ so that

$$|s_n - s'_n| < \varepsilon.$$

Hence s'_n converges to the same limit as $(s_n)_{n\in\mathbb{N}}$. □