Lecture 11

Functions and their properties
Cartesian products and Axion of Choice

MATH 411H, FALL 2025

October 9, 2025

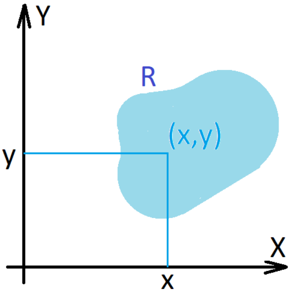
Relations

Relations

A **relation** from X to Y is a subset R of $X \times Y$, i.e. $R \subseteq X \times Y$.

If $X = Y$ we speak about relations on X.

If R is a relation from X to Y we shall sometimes write xRy to mean that $(x, y) \in R \subseteq X \times Y$.



Functions

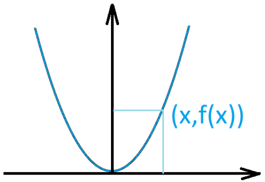
Functions

A **function** $f : X \to Y$ is a relation from X to Y with the property that for every $x \in X$ there is a unique element $y \in Y$ such that xRy in which case we write

$$y = f(x).$$

Example 1

$X = \mathbb{R}$, $f(x) = x^2$.

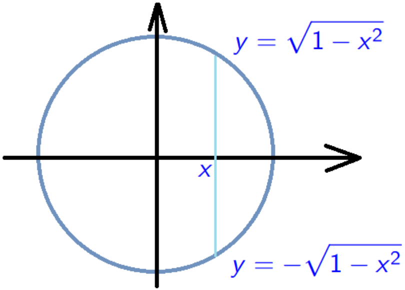


A relation which is not a function

Example 2

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}.$$

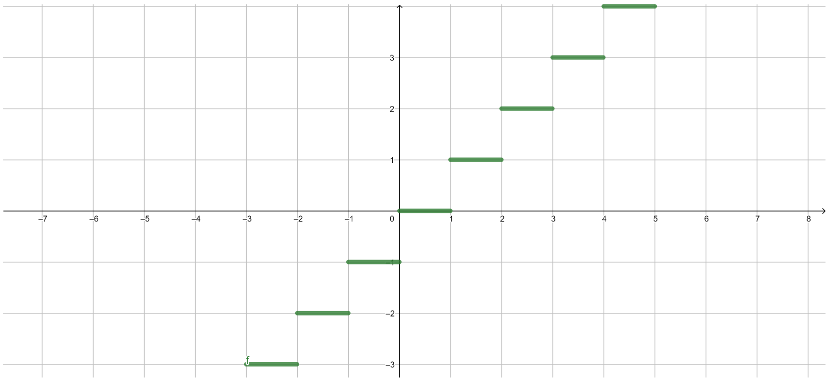
This is not a function!



Examples of functions - integer part

Integer part

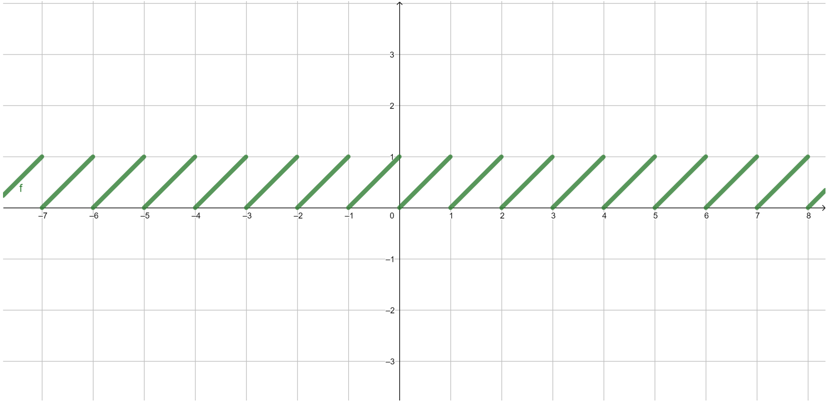
$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}.$$



Examples of functions - fractional part

Fractional part

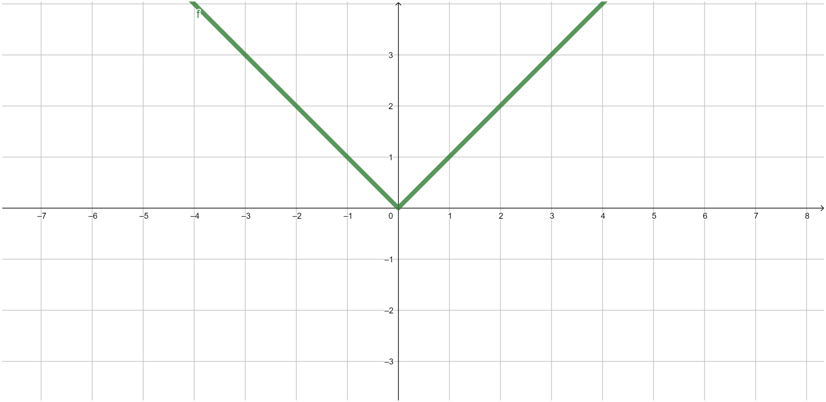
$$\{x\} = x - \lfloor x \rfloor.$$



Examples of functions - absolute value

Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}.$$



Composition of functions

Composition of functions

If $f : X \to Y$ and $g : Y \to Z$ are functions we define their **composition** $g \circ f : X \to Z$ by setting

$$g \circ f(x) = g(f(x)) \quad \text{for} \quad x \in X.$$

Example

If $f(x) = x^2 - 2$ and $g(x) = |x|$, then $g \circ f(x) = |x^2 - 2|$.

Composition of the functions - example

$$f(x) = x^2 - 2, \qquad g(x) = |x|,$$
$$g \circ f(x) = |x^2 - 2|$$

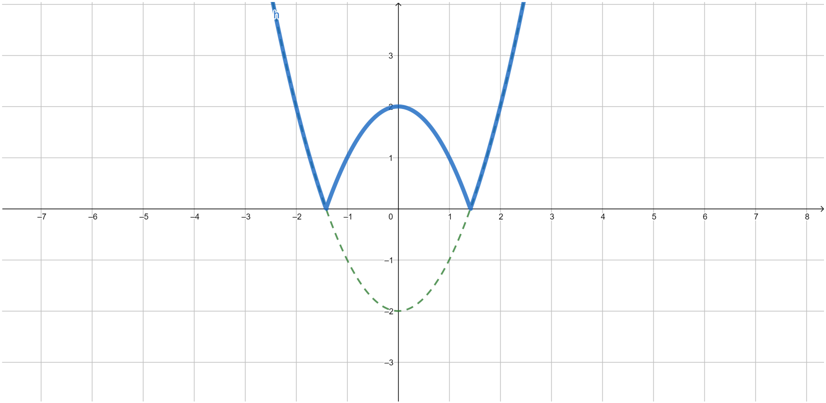


Image and inverse image

Image

If $D \subseteq X$, we define **the image** of D under the function $f : X \to Y$ by

$$f[D] = \{f(x) \ : \ x \in D\}.$$

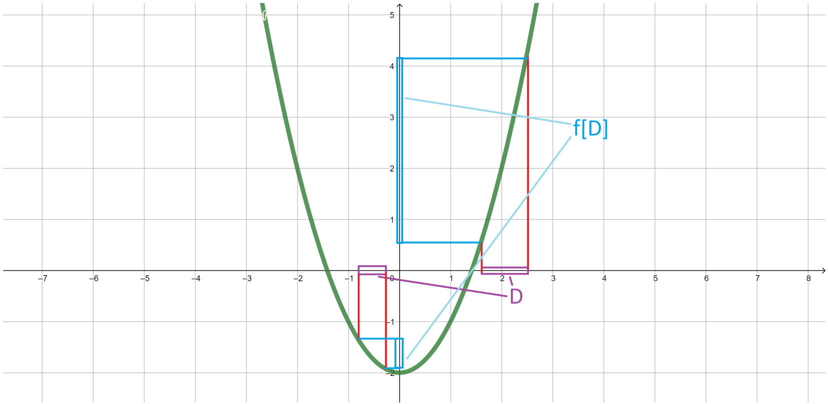
Inverse image

If $E \subseteq Y$, we define **the inverse image** of E under the function $f : X \to Y$ by

$$f^{-1}[E] = \{x \in X \ : \ f(x) \in E\}.$$

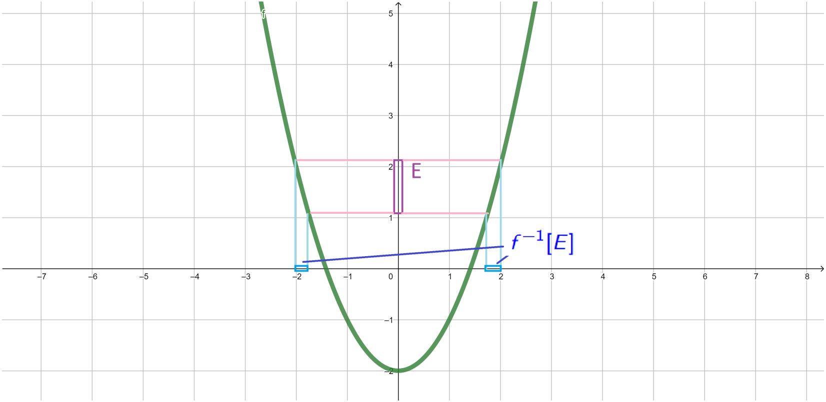
Image - example

$$f[D] = \{f(x) \ : \ x \in D\}.$$



Inverse image - example

$$f^{-1}[E] = \{x \in X \ : \ f(x) \in E\}.$$



Inverse image - properties

For every function $f : X \to Y$ one has

1. $$f^{-1}\left[\bigcup_{\alpha\in A} E_\alpha\right] = \bigcup_{\alpha\in A} f^{-1}[E_\alpha],$$

2. $$f^{-1}\left[\bigcap_{\alpha\in A} E_\alpha\right] = \bigcap_{\alpha\in A} f^{-1}[E_\alpha],$$

3. $$f^{-1}[E^c] = (f^{-1}[E])^c.$$

Image - properties

For every function $f : X \to Y$ one has

$$f\left[\bigcup_{\alpha\in A} E_\alpha\right] = \bigcup_{\alpha\in A} f[E_\alpha].$$

Exercise

Corresponding formulas for intersection and complements may not be true.

Domain and range

Domain

If $f : X \to Y$ is a function, X is called **the domain** of f and denoted by

$$\mathrm{dom}(f) = X.$$

Range

If $f : X \to Y$ is a function, $f[X]$ is called **the range** of f denoted by

$$\mathrm{rgn}(f) = f[X].$$

Example

If $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x^2$, then

$$\mathrm{dom}(f) = \mathbb{R} \qquad \text{and} \qquad \mathrm{rgn}(f) = [0, \infty).$$

Injective functions, 1/2

Injective functions

The function f is said to be **injective** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 1

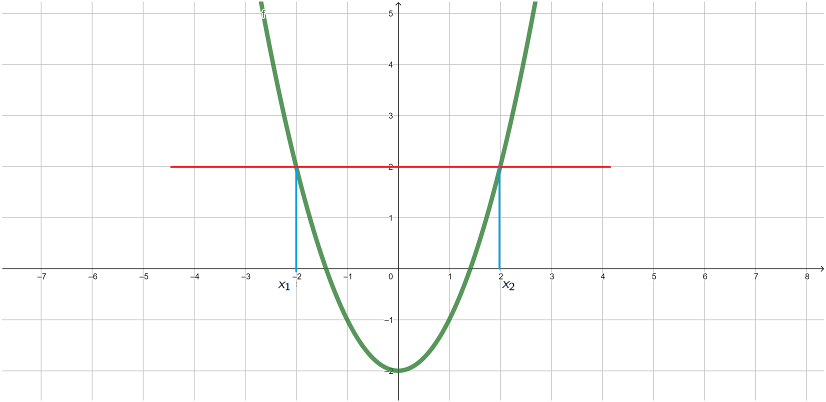
$f(x) = x^2 - 2$ is not injective since for $x_1 = -2$ and $x_2 = 2$ we have

$$f(x_1) = f(-2) = (-2)^2 - 2 = 2,$$

$$f(x_2) = f(2) = 2^2 - 2 = 2.$$

Injective functions - example 1/2

$f(x) = x^2 - 2$



Injective functions, 2/2

Example 2

$f(x) = x^3 + 4$ is injective.

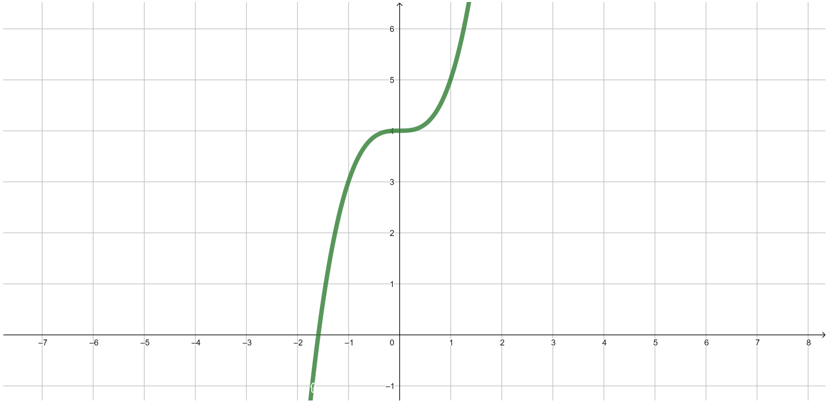
Proof: Indeed, suppose that $f(x_1) = f(x_2)$, then

$$\begin{aligned} f(x_1) = f(x_2) &\iff \\ x_1^3 + 4 = x_2^3 + 4 &\iff \\ x_1^3 = x_2^3 &\iff x_1 = x_2. \end{aligned}$$

□

Injective functions - example 2/2

$f(x) = x^3 + 4$



Surjective functions

Surjective functions

$f : X \to Y$ is said to be **surjective** if $f[X] = Y$.

Example 1

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 5$ is surjective.

Example 2

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 2$ is not surjective since

$$f[\mathbb{R}] = [-2, +\infty) \neq \mathbb{R}.$$

Example 3

Every mapping $f : X \to Y$ is surjective if $Y = f[X]$.

Bijective functions

Bijective functions

$f : X \to Y$ is **bijective** if it is both injective and surjective.

Example 1

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax + b$ is bijective is $a \neq 0$.

Example 2

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 5$ is bijective.

Example 3

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$ is not bijective since it is not injective.

Inverse functions

Inverse functions

If $f : X \to Y$ is bijective it has an inverse $f^{-1} : Y \to X$ such that

$$f^{-1} \circ f \quad \text{and} \quad f \circ f^{-1}$$

are both identity functions, i.e.

$$f^{-1} \circ f(x) = x \quad \text{and} \quad f \circ f^{-1}(y) = y \quad \text{for all} \quad x \in X, y \in Y.$$

Example 1

If $a \neq 0$, then $f(x) = ax + b$ has an inverse given by

$$f^{-1}(x) = \frac{x - b}{a}.$$

Restriction of the function

Restriction

If $A \subseteq X$ we denote $f_{|A}$ **the restriction** of $f : X \to Y$ to A:

$$f_{|A} : A \to Y, \qquad f_{|A}(x) = f(x) \qquad \text{for all} \qquad x \in A.$$

Example 1

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Let $A = [0, +\infty)$ and let $g(x) = f_{|A}$. Then f is not injective, but g is injective.

Set of maps

Cartesian product

If $(X_\alpha)_{\alpha \in A}$ is an indexed family of sets, their Cartesian product

$$\prod_{\alpha \in A} X_\alpha$$

is the set of all maps $f : A \to \bigcup_{\alpha \in A} X_\alpha$ so that $f(\alpha) \in X_\alpha$ for all $\alpha \in A$.

Projection map

If $X = \prod_{\alpha \in A} X_\alpha$ and $\alpha \in A$ we define α**-th projection** or **coordinate map** $\pi_\alpha : X \to X_\alpha$ by $\pi_\alpha(f) = f(\alpha)$.

We will also write $x = (x_\alpha)_{\alpha \in A} \in X = \prod_{\alpha \in A} X_\alpha$ instead of f, and x_α instead of $f(\alpha)$.

Set of maps - examples 1/2

Example 1

If $A = \{1, 2, \ldots, n\}$, then

$$X = \prod_{j=1}^{n} X_j = X_1 \times X_2 \times \ldots \times X_n$$
$$= \{(x_1, x_2, \ldots, x_n) \;:\; x_j \in X_j \text{ for all } j = 1, 2, \ldots, n\}.$$

Example 2

If $A = \mathbb{N}$, then

$$X = \prod_{j=1}^{\infty} X_j = X_1 \times X_2 \times \ldots = \{(x_1, x_2, \ldots) \;:\; x_j \in X_j \text{ for all } j = 1, 2, \ldots\}.$$

Set of maps - examples 2/2

Example 3

If the sets X_α are all equal to some fixed set Y, we denote

$$X = \prod_{\alpha \in A} X_\alpha = Y^A.$$

Y^A-the set of all mappings from A to Y.

Example 4

$\mathbb{Z}^{\mathbb{N}}$ - the set of all sequences of integers.

$\mathbb{R}^{\mathbb{N}}$ - the set of all sequences of real numbers.

The axiom of choice

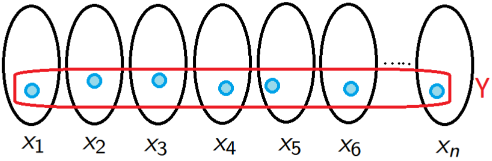
The axiom of choice

If $(X_\alpha)_{\alpha\in A}$ is a nonempty collection of nonempty sets, then

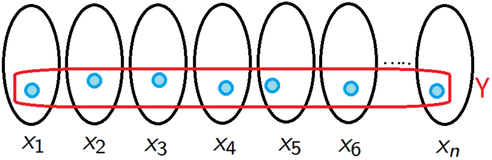
$$\prod_{\alpha\in A} X_\alpha \neq \emptyset.$$

Corollary

If $(X_\alpha)_{\alpha\in A}$ is a disjoint collection of nonempty sets, then there is a set $Y \subseteq \bigcup_{\alpha\in A} X_\alpha$ (called **the selector** of $(X_\alpha)_{\alpha\in A}$) such that $Y \cap X_\alpha$ contains precisely one element for each $\alpha \in A$.



Proof of Corollary



Take $f \in \prod_{\alpha \in A} X_\alpha \neq \emptyset$. Define

$$Y = f[A],$$

then

$$Y \cap X_\alpha = \{f(\alpha)\},$$

since $f(\alpha) \in X_\alpha$. □

Remark

In fact, this corollary is equivalent to the axiom of choice. Why?