# Lecture 13

## Countable sets, cardinality continuum

MATH 411H, FALL 2025

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# Cardinality

## Cardinality

If $X$ and $Y$ are nonempty sets, we define the expressions

$$\mathrm{card}\,(X) \leq \mathrm{card}\,(Y) \quad \text{(injective)},$$

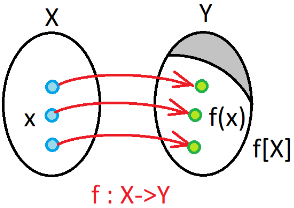
$$\mathrm{card}\,(X) = \mathrm{card}\,(Y) \quad \text{(bijective)},$$

$$\mathrm{card}\,(X) \geq \mathrm{card}\,(Y) \quad \text{(surjective)},$$

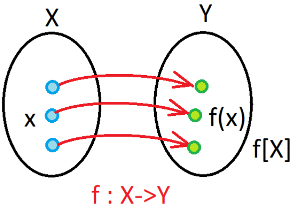
to mean that there exists $f : X \to Y$ which is injective, bijective, surjective respectively.

# Cardinality - pictures 1/2

$\text{card}\,(X) \leq \text{card}\,(Y)$, (injective),

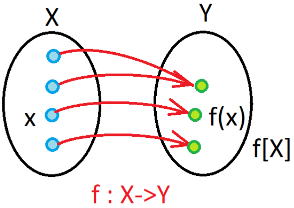



$\text{card}\,(X) = \text{card}\,(Y)$, (bijective)



# Cardinality - pictures 2/2

$\text{card}(X) \geq \text{card}(Y)$, (surjective)




- We also define $\text{card}(X) < \text{card}(Y)$ to mean that there is an injection but not a bijection.
- We also have $\text{card}(\emptyset) < \text{card}(X)$ and $\text{card}(X) > \text{card}(\emptyset)$ for all $X \neq \emptyset$.

# $\mathrm{card}\,(X)$ -example

**Example**

Let $X = \{1, 2, 3, 4, \ldots\}$ and $Y = \{101, 102, 103, \ldots,\}$. Prove that $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

**Solution.** Let us define $f : X \to Y$ by $f(x) = x + 100$, then $f$ is a bijection between $X$ and $Y$, so $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □

**Example**

Let $X = \{1, 2, 3, 4, \ldots\}$ and $Y = \{1^2, 2^2, 3^2, 4^2, \ldots,\}$. Prove that $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

**Solution.** Let us define $f : X \to Y$ by

$$f(x) = x^2,$$

then $f$ is a bijection between $X$ and $Y$, so $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □

# $\mathrm{card}\,(X)$ - example

**Example**

Let $X = \{1, 2, 3\}$ and $Y = \{2, 4, 6, 8\}$. Prove that $\mathrm{card}\,(X) < \mathrm{card}\,(Y)$.

**Solution.** Note that $f(x) = 2x$ is an injection from $X$ to $Y$, so $\mathrm{card}\,(X) \leq \mathrm{card}\,(Y)$. On the other hand, any function from $X$ to $Y$ takes at most 3 values, so it is not a surjection, so $\mathrm{card}\,(X) < \mathrm{card}\,(Y)$. □

**Example**

Let $X = [0, 1]$ and $Y = [1, 3]$. Prove that $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

**Solution.** Let us define $f : X \to Y$ by

$$f(x) = 2x + 1,$$

then $f$ is a bijection between $X$ and $Y$, so $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □

# Proposition

**Proposition**

We have

$$\mathrm{card}\,(X) \leq \mathrm{card}\,(Y) \iff \mathrm{card}\,(Y) \geq \mathrm{card}\,(X).$$

**Proof ($\Rightarrow$).** Assume that $\mathrm{card}\,(X) \leq \mathrm{card}\,(Y)$. This means that there is an injection $f : X \to Y$. Thus $f$ is a bijection $f : X \to f[X] \subseteq Y$. Let $f^{-1}$ be the inverse $f^{-1} : f[X] \to X$. Pick $x_0 \in X$ and define $g$ by

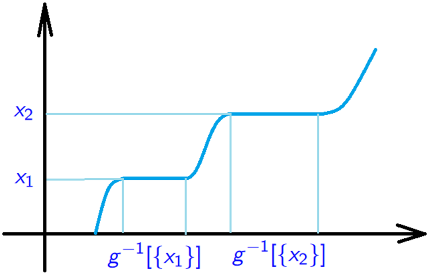
$$g(y) = \begin{cases} f^{-1}(y) \text{ if } y \in f[X], \\ g(y) = x_0 \text{ if } y \in Y \setminus f[X]. \end{cases}$$

Then we see that $g$ is surjective from $Y$ to $X$.

# Proof: 1/2

**Proof (⇐).** If $\mathrm{card}\,(Y) \geq \mathrm{card}\,(X)$, then there is a surjection $g : Y \to X$. Then $g[Y] = X$, and, consequently, $g^{-1}[\{x\}]$ are nonempty and

$$g^{-1}[\{x_1\}] \cap g^{-1}[\{x_2\}] = \emptyset \quad \text{if} \quad x_1 \neq x_2.$$



# Proof: 2/2

Using the axiom of choice the set $\prod_{x\in X} g^{-1}[\{x\}] \neq \emptyset$. Taking

$$f \in \prod_{x\in X} g^{-1}[\{x\}]$$

we see that $f$ is an injection from $X$ to $Y$. Indeed, if $x_1 \neq x_2$, then $f(x_1) \in g^{-1}[\{x_1\}]$ and $f(x_2) \in g^{-1}[\{x_2\}]$, but

$$g^{-1}[\{x_1\}] \cap g^{-1}[\{x_2\}] = \emptyset,$$

thus $f(x_1) \neq f(x_2)$. $\square$

# $\operatorname{card}(X)$ - example

**Example**

Let

$$X = [0,1],$$

$$Y = \left\{ \frac{1}{n} \ : \ n \in \mathbb{N} \right\}.$$

Prove that $\operatorname{card}(X) \geq \operatorname{card}(Y)$.

**Solution.** One has $\operatorname{card}(Y) \leq \operatorname{card}(X)$. Indeed, define $f : Y \to X$ by

$$f(x) = x,$$

which is injective, so $\operatorname{card}(X) \geq \operatorname{card}(Y)$. $\square$

**Remark**

Actually we have $\operatorname{card}(X) > \operatorname{card}(Y)$, which will be proved later.

# Theorem

**Theorem**

For any sets $X$ and $Y$

$$\text{either} \quad \mathrm{card}\,(X) \leq \mathrm{card}\,(Y) \quad \text{or} \quad \mathrm{card}\,(Y) \leq \mathrm{card}\,(X).$$

**Proof.** Consider

$$\mathcal{I} = \{f : \overbrace{A}^{\text{dom}} \to \overbrace{B}^{\text{rng}} : f \text{ is injection and } A \subseteq X,\ B \subseteq Y\}$$

We say $f \preccurlyeq g$ for any $f, g \in \mathcal{I}$ if

$$\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \quad \text{and} \quad f(x) = g(x) \ \text{ if } \ x \in \mathrm{dom}(f).$$

- It is easy to see that $(\mathcal{I}, \preccurlyeq)$ is a poset. Moreover, $(\mathcal{I}, \preccurlyeq)$ satisfies the hypotheses of Kuratowski–Zorn's lemma.

# Proof

- By Kuratowski–Zorn's lemma, $\mathcal{I}$ has a maximal element $f \in \mathcal{I}$ with $\mathrm{dom}(f) = A$ and $\mathrm{rng}(f) = B$.
- By maximality we deduce that $A = X$ or $Y = B$. If not, there are $x_0 \in X \setminus A$ and $y_0 \in Y \setminus B$, then $f$ can be extended to a one-to-one map from $A \cup \{x_0\}$ to $B \cup \{y_0\}$. We simply set

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ y_0 & \text{if } x = x_0. \end{cases}$$

- Then $\widetilde{f}$ is an extension of $f$, but this is a contradiction since $f$ is a maximal element in $\mathcal{I}$. Hence either

$$A = X, \quad \text{then} \quad \mathrm{card}\,(X) \leq \mathrm{card}\,(Y),$$

or

$$B = Y, \quad \text{then} \quad \mathrm{card}\,(Y) \leq \mathrm{card}\,(X),$$

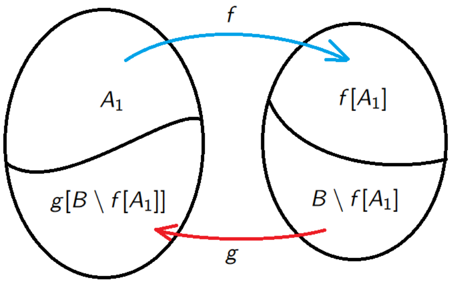
and the proof is finished. □

# Banach lemma

## Banach lemma

Let $f : A \to B$ and $g : B \to A$ be injections. Then there are sets $A_1, A_2, B_1, B_2$ such that

1. $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$,
2. $B_1 \cup B_2 = B$ and $B_1 \cap B_2 = \emptyset$,
3. $f[A_1] = B_1$ and $f[A_2] = B_2$.



# Proof

**Proof.** Consider the map $\Phi : P(A) \to P(A)$ defined by

$$\Phi(X) = A \setminus g[B \setminus f[X]] \quad \text{for} \quad X \in P(X).$$

Since $g$ is injective we note that

$$\Phi\left[\bigcup_{t\in I} A_t\right] = \bigcup_{t\in I} \Phi[A_t]$$

for any family $(A_t)_{t\in I}$ such that $A_t \subseteq A$ for all $t \in I$. Consider

$$\Omega = \emptyset \cup \Phi[\emptyset] \cup \Phi \circ \Phi[\emptyset] \cup \ldots \cup \overbrace{\Phi \circ \Phi \circ \ldots \circ \Phi}^{n}[\emptyset] \cup \ldots,$$

in other words

$$\Omega = \bigcup_{n=0}^{\infty} \Phi^n[\emptyset].$$

# Proof

Then

$$\Phi[\Omega] = \bigcup_{n=0}^{\infty} \Phi^{n+1}[\emptyset] = \bigcup_{n=1}^{\infty} \Phi^{n}[\emptyset] = \left(\bigcup_{n=1}^{\infty} \Phi^{n}[\emptyset]\right) \cup \emptyset = \bigcup_{n=0}^{\infty} \Phi^{n}[\emptyset] = \Omega,$$

thus the set $\Omega$ if a fixed point of $\Phi$, i.e. $\Phi[\Omega] = \Omega$. Taking

$$A_1 = \Omega, \quad A_2 = A \setminus A_1,$$

$$B_1 = f[A_1], \quad B_2 = B \setminus B_1$$

we obtain

$$A_1 = \Phi[A_1] = A \setminus g[B \setminus f[A_1]] = A \setminus g[B \setminus B_1] = A \setminus g[B_2],$$

thus

$$A_2 = A \setminus A_1 = A \setminus (A \setminus g[B_2]) = g[B_2].$$

# Cantor–Bernstein–Schröder theorem

**Cantor–Bernstein–Schröder theorem**

If $\mathrm{card}\,(X) \leq \mathrm{card}\,(Y)$ and $\mathrm{card}\,(Y) \leq \mathrm{card}\,(X)$, then $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

**Proof.** By Banach lemma there are $A_1, A_2, B_1, B_2$ such that

$$A_1 \cup A_2 = X, \quad f[A_1] = B_1, \quad A_1 \cap A_2 = \emptyset$$

$$B_1 \cup B_2 = X, \quad g[B_2] = A_2, \quad B_1 \cap B_2 = \emptyset$$

whenever $f : X \to Y$ and $g : Y \to X$ are injections. Define $h : X \to Y$ by setting

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_1, \\ g^{-1}(x) & \text{if } x \in A_2. \end{cases}$$

Then we see that $h$ is a bijection between $X$ and $Y$. □

# Cantor–Bernstein–Schröder theorem - example

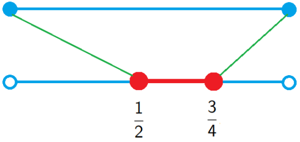
## Example

Let $X = [0,1], \quad Y = (0,1)$. Prove that $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$.

**Solution.** Note that $\mathrm{card}\,(Y) \leq \mathrm{card}\,(X)$, because $f(x) = x$ is injective. Then, let us prove that $\mathrm{card}\,(X) \leq \mathrm{card}\,(Y)$. We define

$$g(x) = \frac{x}{4} + \frac{1}{2}.$$

One can check that $g$ is injective, so, by **Cantor–Bernstein–Schröder theorem**, $\mathrm{card}\,(X) = \mathrm{card}\,(Y)$. □



# Cantor's theorem

**Cantor's Theorem**

For any set $X$ we have $\mathrm{card}\,(X) < \mathrm{card}\,(P(X))$.

**Proof.** The map $f : X \to P(X)$ defined by $f(x) = \{x\}$ is an injection from $X$ to $P(X)$. Thus $\mathrm{card}\,(X) \leq \mathrm{card}\,(P(X))$.

We now show that there is no bijection between $X$ and $P(X)$. Let $g : X \to P(X)$ and consider the set

$$Y = \{x \in X \ : \ x \notin g(x)\} \in P(X).$$

Then we claim that $Y \notin g[X]$.

# Proof

If

$$Y = \{x \in X \ : \ x \notin g(x)\} \in g[X],$$

then there is $x_0 \in X$ so that $g(x_0) = Y$.

- On the one hand
$$x_0 \in Y \iff x_0 \notin g(x_0).$$
- On the other hand
$$x_0 \in Y \iff x_0 \in g(x_0).$$
- Thus
$$x_0 \in g(x_0) \iff x_0 \notin g(x_0),$$
which is impossible, and we conclude $\mathrm{card}\,(X) < \mathrm{card}\,(P(X))$. □

# Countable set

## Countable set

A set $X$ is called **countable (or denumerable)** if

$$\operatorname{card}(X) \leq \operatorname{card}(\mathbb{N}).$$

### Example 1

In particular, finite sets are countable and for these sets it is convenient to interpret $\operatorname{card}(X)$ as the number of elements in $X$:

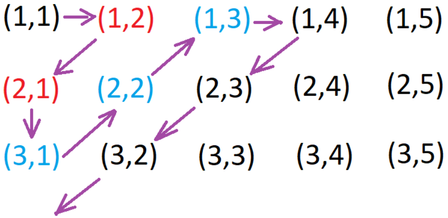
$$\operatorname{card}(X) = n \iff \operatorname{card}(X) = \operatorname{card}(\{1, 2, \ldots, n\}).$$

### Example 2

If $X$ is countable but not finite, we say that $X$ is **countably infinite**.

# Countable sets - example

The set $\mathbb{N} \times \mathbb{N}$ is countable, i.e. $\text{card}\,(\mathbb{N} \times \mathbb{N}) = \text{card}\,(\mathbb{N})$. Note that




can be listed as a sequence

$(1,1), (1,2), (2,1), (1,3), (2,2), (3,1), (1,4), (2,3), (3,2), (4,1), (5,1), (4,2), \ldots$

establishing a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

# Proposition

## Proposition

- (a) If $X$ and $Y$ are countable, so is $X \times Y$.
- (b) If $A$ is countable and $X_\alpha$ is countable for all $\alpha \in A$, then
$$\bigcup_{\alpha \in A} X_\alpha \text{ is countable.}$$
- (c) If $X$ is countably infinite then $\operatorname{card}(X) = \operatorname{card}(\mathbb{N})$.

**Proof of (a).** To prove (a) it suffices to show that $\operatorname{card}(\mathbb{N} \times \mathbb{N}) = \operatorname{card}(\mathbb{N})$, but it was shown in the previous example.

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**Proof of (b).** For each $\alpha \in A$ there is a surjection $f_\alpha : \mathbb{N} \to \mathbb{X}_\alpha$ (here we have used the axiom of choice).

# Proof of Proposition 1/2

Then the map $f : \mathbb{N} \times \mathbb{N} \to \bigcup_{\alpha \in A} X_\alpha$ defined by

$$f(n, \alpha) = f_\alpha(n)$$

is surjective and we are done. Alternatively, to prove

$$\mathrm{card}\left(\bigcup_{\alpha \in A} X_\alpha\right) \leq \mathrm{card}\,(\mathbb{N}),$$

we can also proceed as follows

$$X_1 = \{x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, \ldots\}$$
$$X_2 = \{x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, \ldots\}$$
$$X_3 = \{x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4}, \ldots\}$$
$$\ldots$$
$$\bigcup_{j \in \mathbb{N}} X_j = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{1,3}, x_{2,2}, x_{3,1}, \ldots\}.$$

# Proof of Proposition 2/2

**Proof of (c).** We can assume that $X$ is an infinite subset of $\mathbb{N}$.

Let $f(1)$ be the smallest element of $X$ and define inductively

$$f(n) = \min\left\{X \setminus \{f(1), f(2), \dots, f(n-1)\}\right\}.$$

Then it can be easily verified that $f$ is a bijection from $\mathbb{N}$ to $X$. □

**Corollary**

$\mathbb{Z}$ and $\mathbb{Q}$ are countable.

**Proof.** One has $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-n \ : \ n \in \mathbb{N}\}$ and

$$\mathbb{Q} = \bigcup_{n\in\mathbb{N}} \{\frac{m}{n} \ : \ m \in \mathbb{Z}\}.$$

Note also that

$$\mathrm{card}\,(\mathbb{Q}) = \mathrm{card}\,(\mathbb{N}) = \mathrm{card}\,(\mathbb{N} \cup \{0\}) = \mathrm{card}\,(\mathbb{Z}).$$ □

# Countable sets - example

## Example

Prove that the following set is countable $X = \{(n, m, k) \ : \ n, m, k \in \mathbb{N}\}$.

**Solution.** Fix $k \in \mathbb{N}$. It is clear that $f(n, m) = (n, m, k)$ is a bijection between $\mathbb{N} \times \mathbb{N}$ and $A_k = \{(n, m, k) \ : \ n, m \in \mathbb{N}\}$, so $A_k$ is countable. Then we write $X = \bigcup_{k \in \mathbb{N}} A_k$ and use the previous theorem. □

## Example

Prove that the set $\mathbb{Q} \times \mathbb{Q}$ is countable.

**Solution.** Fix $q \in \mathbb{Q}$. It is easy to check that $f(x) = (x, q)$ is bijection between $\mathbb{Q}$ and $A_q = \{(r, q) \ : \ r \in \mathbb{Q}\}$. Hence $A_q$ is countable. We write

$$\mathbb{Q} \times \mathbb{Q} = \bigcup_{q \in \mathbb{Q}} A_q$$

and use the previous theorem. □

# Proposition

## Proposition

$\{0,1\}^{\mathbb{N}_0}$ is uncountable.

**Proof.** Suppose that the set $\{0,1\}^{\mathbb{N}_0}$ is countable, then

$$\{0,1\}^{\mathbb{N}_0} = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}.$$

$$\begin{aligned}
\alpha_0 :& \quad \alpha_0(0), \alpha_0(1), \alpha_0(2), \ldots \\
\alpha_1 :& \quad \alpha_1(0), \alpha_1(1), \alpha_1(2), \ldots \\
\alpha_2 :& \quad \alpha_2(0), \alpha_2(1), \alpha_2(2), \ldots \\
& \quad \ldots
\end{aligned}$$

consider a new sequence $\Delta : \mathbb{N}_0 \to \{0,1\}$ defined by $\Delta(n) = 1 - \alpha_n(n)$. Then $\Delta \neq \alpha_n$ for all $n \in \mathbb{N}_0$, thus $\Delta \notin \{\alpha_n \ : \ n \in \mathbb{N}_0\}$, but $\Delta \in \{0,1\}^{\mathbb{N}_0}$, contradiction. $\square$

# Cardinality continuum

## Cardinality continuum

A set $X$ is said to have cardinality **continuum** if

$$\mathrm{card}\,(X) = \mathrm{card}\,(\mathbb{R}).$$

We shall write $\mathrm{card}\,(X) = \mathfrak{c}$ iff $\mathrm{card}\,(X) = \mathrm{card}\,(\mathbb{R})$.

## Theorem

$$\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}) = \mathrm{card}\,(\mathbb{R}).$$

# Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}})$: 1/2

We first show that

$$\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}).$$

Define $F : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ by setting

$$F(A) = \chi_A \in \{0,1\}^{\mathbb{N}},$$

$$\chi_A(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

Example

If $A = \{1,2,3,5,7,9\}$, then $\chi_A$ can be identified with the sequence

$$(\underbrace{1}_{1}, \underbrace{1}_{2}, \underbrace{1}_{3}, \underbrace{0}_{4}, \underbrace{1}_{5}, \underbrace{0}_{6}, \underbrace{1}_{7}, \underbrace{0}_{8}, \underbrace{1}_{9}, \underbrace{0}_{10}, \underbrace{0}_{11}, \ldots)$$

# Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}})$: 2/2

Our aim is to show that $F$ is a bijection.

- Let $A, B \subseteq \mathbb{N}$ be such that $A \neq B$, then we show that $F(A) \neq F(B)$. Since $A \neq B$ (wlog) there exists $x_0 \in A \setminus B$, thus

$$1 = \chi_A(x_0) \neq \chi_B(x_0) = 0 \quad \Longleftrightarrow \quad F(A) \neq F(B),$$

  which proves that $F : P(\mathbb{N}) \to \{0,1\}^{\mathbb{N}}$ is injective.

- To show that $F$ is surjective take $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ and consider

$$E = \{n \in \mathbb{N} \ : \ \alpha_n = 1\}.$$

  then

$$F(E) = \chi_E = \alpha \quad \Longleftrightarrow \quad 1 = \alpha_n = \chi_E(n) = 1.$$

# Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R})$: 1/3

- It remains to prove that

$$\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R}).$$

- We first show that $\mathrm{card}\,(P(\mathbb{N})) \leq \mathrm{card}\,(\mathbb{R})$.
- We now construct an injection $f : P(\mathbb{N}_0) \to \mathbb{R}$ by setting

$$f(A) = \sum_{n \in A} \frac{2}{3^{n+1}} \quad \text{for any} \quad A \subseteq \mathbb{N}_0.$$

- $f(\mathbb{N}_0) = 1$, since

$$\sum_{n=0}^{N} \frac{2}{3^{n+1}} = \frac{2}{3} \frac{1 - \frac{1}{3^{N+1}}}{1 - \frac{1}{3}} = 1 - \frac{1}{3^{N+1}} \leq 1.$$

# Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R})$: 2/3

Now we show that $f : P(\mathbb{N}_0) \to \mathbb{R}$ is injective. Let $A, B \subseteq \mathbb{N}$ be such that $A \neq B$. Let

$$n_0 = \min\{n \in \mathbb{N}_0 \colon \chi_A(n) \neq \chi_B(n)\}.$$

Wlog we can assume $n_0 \in B \setminus A$. Then

$$\begin{aligned} f(A) = \sum_{n\in A} \frac{2}{3^{n+1}} &= \sum_{\substack{n\in A\\ n<n_0}} \frac{2}{3^{n+1}} + \sum_{\substack{n\in A\\ n>n_0}} \frac{2}{3^{n+1}} \\ &\leq \sum_{\substack{n\in B\\ n<n_0}} \frac{2}{3^{n+1}} + \sum_{n\in(n_0,\infty)} \frac{2}{3^{n+1}} \\ &< \sum_{\substack{n\in B\\ n<n_0}} \frac{2}{3^{n+1}} + \frac{2}{3^{n_0+1}} \leq f(B). \end{aligned}$$

# Proof of $\mathrm{card}\,(P(\mathbb{N})) = \mathrm{card}\,(\mathbb{R})$: 3/3

- We have used $f((n_0, \infty) \cap \mathbb{N}_0) = \frac{1}{3^{n_0+1}} < \frac{2}{3^{n_0+1}}$ (check this!). Hence $f(A) \neq f(B)$ as desired, yielding $\mathrm{card}\,(P(\mathbb{N}_0)) \leq \mathrm{card}\,(\mathbb{R})$.
- Since $\mathrm{card}\,(\mathbb{N}_0) = \mathrm{card}\,(\mathbb{Q})$ thus $\mathrm{card}\,(P(\mathbb{N}_0)) = \mathrm{card}\,(P(\mathbb{Q}))$. Now define $g : \mathbb{R} \to P(\mathbb{Q})$ by

$$g(x) = \{r \in \mathbb{Q} \colon r < x\} \quad \text{for any} \quad x \in \mathbb{R}.$$

It is easy to see that $g$ is injective, thus

$$\mathrm{card}\,(\mathbb{R}) \leq \mathrm{card}\,(P(\mathbb{Q})) = \mathrm{card}\,(P(\mathbb{N}_0)).$$

By the **Cantor–Bernstein–Schröder** theorem

$$\mathrm{card}\,(\mathbb{R}) = \mathrm{card}\,(P(\mathbb{N}_0)) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}). \quad \square$$

# Remarks

- **A** If $\operatorname{card}(X) \geq \mathfrak{c}$, then $X$ is uncountable.
- **B** $\operatorname{card}(\mathbb{R} \times \mathbb{R}) = \operatorname{card}(\mathbb{R})$.
- **C** $\operatorname{card}(\{0,1\}^{\mathbb{N}_0} \times \{0,1\}^{\mathbb{N}_0}) = \operatorname{card}(\{0,1\}^{\mathbb{N}_0})$.
- **D** $\operatorname{card}(\mathbb{R}^k) = \operatorname{card}(\mathbb{R})$ for any $k \in \mathbb{N}$.
- **E** If $\operatorname{card}(X) \leq \mathfrak{c}$ and $\operatorname{card}(Y) \leq \mathfrak{c}$, then $\operatorname{card}(X \times Y) \leq \mathfrak{c}$.
- **F** If $\operatorname{card}(A) \leq \mathfrak{c}$ and $\operatorname{card}(X_\alpha) \leq \mathfrak{c}$ for any $\alpha \in A$, then

$$\operatorname{card}\left(\bigcup_{\alpha \in A} X_\alpha\right) \leq \mathfrak{c}.$$

# Uncountable sets - example

## Example

Prove that $\mathrm{card}\,(\mathbb{R}) = \mathrm{card}\,([0,1])$.

**Solution.** Define $f : \mathbb{R} \to [0,1]$ by $f(x) = \frac{x}{|x|+1}$. One can verify that $f$ is a bijection between $\mathbb{R}$ and $[0,1]$. □

## Example

Determine $\mathrm{card}\,(X)$, where $X$ is

$$\{[n, n+1) \ : \ n \in \mathbb{N}\}.$$

**Solution.** We will prove that $\mathrm{card}\,(X) = \mathrm{card}\,(\mathbb{N})$. Define

$$f([n, n+1)) = n.$$

It is easy to verify that $f$ is bijection between $\mathbb{N}$ and $X$. □

# Uncountable sets - example

Example

Determine $\mathrm{card}\,(X)$, where $X$ is any infinite set of pairwise disjoint closed intervals.

**Solution.** We will prove that $X$ is **countably infinite**. Let us define the function $f : X \to \mathbb{Q}$ by setting

$$f(I) = q,$$

where $q$ is any rational number contained in interval $I \in X$. Then $f$ is injective, so $\mathrm{card}\,(X) \leq \mathrm{card}\,(\mathbb{N})$. □