Lecture 14

Metric spaces basic properties

MATH 411H, FALL 2025

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Metric spaces

Metric

A metric on a set X is a function $\rho : X \times X \to [0, \infty)$ such that

1. (i) $\rho(x, y) = 0$ iff $x = y$,
2. (ii) $\rho(x, y) = \rho(y, x)$ for all $x, y, \in X$,
3. (iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

The function $\rho(x, y)$ can be identified as the distance from x to y.

Metric space

A set X equipped with a metric ρ is called a **metric space** and denoted by (X, ρ).

Metric spaces - examples

Example 1

The set $\mathbb{R}$ with $\rho(x,y) = |x-y|$ is a metric space. Clearly ρ is a metric:

1. (i) $\rho(x,y) = |x-y| = 0 \iff x-y=0 \iff x=y$.
2. (ii) $\rho(x,y) = |x-y| = |y-x| = \rho(y,x)$,
3. (iii) $\rho(x,y) = |x-y| \leq |x-z| + |z-y| = \rho(x,z) + \rho(z,y)$.

Example 2

If X is any set, then

$$\rho(x,y) = \begin{cases} 0 \text{ if } x \neq y, \\ 1 \text{ if } x = y \end{cases}$$

is **discrete metric** and (X,ρ) is **a discrete space**.

Metric spaces - examples

Example 3

Consider d-dimensional vector space $\mathbb{R}^d = \overbrace{\mathbb{R} \times \ldots \times \mathbb{R}}^{d \text{ times}}$ and for vectors $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ define

$$\rho_2(x, y) = \left(\sum_{j=1}^{d} |x_j - y_j|^2 \right)^{1/2}$$

$$\rho_1(x, y) = \sum_{j=1}^{d} |x_j - y_j|,$$

$$\rho_\infty(x, y) = \max_{1 \le j \le d} |x_j - y_j|.$$

It is not difficult to see that ρ_2, ρ_1, ρ_∞ are metrics on $\mathbb{R}^d$.

Metric spaces - examples

Example 4

Consider infinite dimensional vector space $\mathbb{R}^{\mathbb{N}}$, and for vectors $x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$, $y = (y_1, y_2, \dots) \in \mathbb{R}^{\mathbb{N}}$, define

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, |x_n - y_n|),$$

which is a metric on $\mathbb{R}^{\mathbb{N}}$.

Example 5

If ρ is a metric on X and $A \subseteq X$ then $\rho|_{A \times A}$ is a metric on A.

Metric spaces - examples

Example 6

Let $(X_1, \rho_1), \ldots, (X_d, \rho_d)$ be metric spaces, and consider their product $X = X_1 \times \ldots \times X_d$. For $x = (x_1, \ldots, x_d) \in X$, $y = (y_1, \ldots, y_d) \in X$, define functions

$$d_1(x, y) = \sum_{j=1}^{d} \rho_j(x_j, y_j),$$

$$d_2(x, y) = \left(\sum_{j=1}^{d} \rho_j(x_j, y_j)^2 \right)^{1/2},$$

$$d_\infty(x, y) = \max_{1 \le j \le d} \rho_j(x_j, y_j),$$

which are metrics on X.

Balls

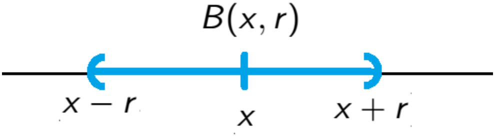
Ball

Let (X, ρ) be a metric space. If $x \in X$ and $r > 0$, **the open ball of radius r and center x** is

$$B(x, r) = \{y \in X \ : \ \rho(x, y) < r\}.$$

Example 1

For $\mathbb{R}$ with $\rho(x, y) = |x - y|$, the ball $B(x, r) = (x - r, x + r)$ is an interval of length $2r$ centered at x:

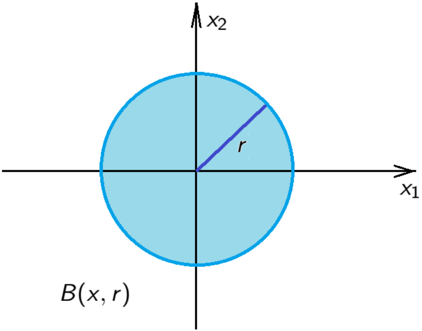


Example 2

$\mathbb{R}^2$ with $\rho_2(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$.

$$\begin{aligned} B(x,r) &= \{y \in \mathbb{R}^2 \ : \ \rho_2(x,y) < r\} \\ &= \{y \in \mathbb{R}^2 \ : \ (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2\} \end{aligned}$$

$$B(0,r) = \{y \in \mathbb{R}^2 \ : \ y_1^2 + y_2^2 < r^2\}$$

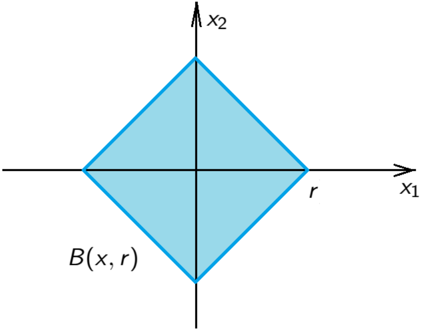


Example 3

$\mathbb{R}^2$ with $\rho_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$, $x = (x_1, x_2)$, $y = (y_1, y_2)$.

$$B(x,r) = \{y \in \mathbb{R}^2 \ : \ \rho_1(x,y) < r\}$$

$$B(0,r) = \{y \in \mathbb{R}^2 \ : \ \rho_1(0,y) < r\}$$

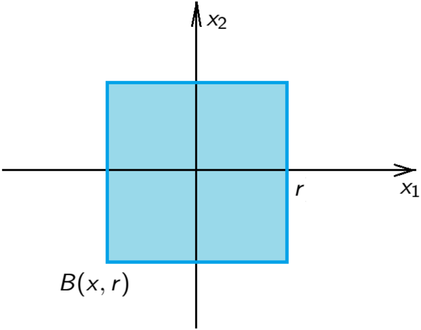


Example 4

$\mathbb{R}^2$ with $\rho_\infty(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$.

$$B(x, r) = \{y \in \mathbb{R}^2 \ : \ \rho_\infty(x, y) < r\}$$

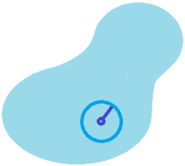
$$B(0, r) = \{y \in \mathbb{R}^2 \ : \ \rho_\infty(0, y) < r\}$$



Open and closed sets

Open set

A set $E \subseteq X$ is **open** if for every $x \in E$ there exists $r > 0$ such that $B(x, r) \subseteq E$.



Closed set

A set $E \subseteq X$ is **closed** if its complement $X \setminus E$ is open.

Open sets - examples

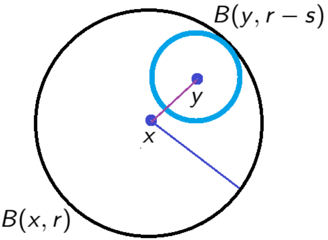
Example 1

Let (X, ρ) be a metric space. Every ball $B(x, r)$ is open, if $y \in B(x, r)$ and $\rho(x, y) = s$, then

$$B(y, r - s) \subseteq B(x, r).$$

Indeed, $z \in B(y, r - s) \iff \rho(y, z) < r - s$, then

$$\rho(x, z) \leq \rho(x, y) + \rho(y, z) < s + r - s = r \iff z \in B(x, r).$$



Open sets - examples

Example 2

X and $\emptyset$ are both open and closed.

Example 3

The union of any family of open sets is open. In other words, if $(U_\alpha)_{\alpha\in A}$ is a family of subsets of X such that each $U_\alpha \subseteq X$ is open, then

$$\bigcup_{\alpha\in A} U_\alpha \text{ is also open.}$$

Open sets - examples

Example 4

The intersection of any finite family of open sets is open.

Indeed, if $U_1, \ldots, U_n$ are open and $x \in \bigcap_{j=1}^n U_j$, then for each $1 \leq j \leq n$ there exists $r_j > 0$ such that

$$B(x, r_j) \subseteq U_j$$

and then

$$B(x, r) \subseteq \bigcap_{j=1}^{n} U_j \quad \text{if} \quad r = \min\{r_1, \ldots, r_n\},$$

so $\bigcap_{j=1}^n U_j$ is open.

Remark

By passing to the complements

1. The intersection of any family of closed sets is closed.
2. The union of any finite family of closed sets is closed.

Example 1

If $x_1, x_2 \in X$, then

$$X \setminus B(x_1, r) \cup X \setminus B(x_2, r)$$

is closed for any $r > 0$.

Interior and closure

Let X be a metric space and let $E \subseteq X$.

Interior of E

The union of all open sets $U \subseteq E$ is the largest open set contained in E and it is called **the interior of** E and it is denoted by

$$\mathrm{int}\, E.$$

Closure of E

The intersecion of all closed sets $F \supseteq E$ is the smallest closed set containing E and it is called **the closure of** E and it is denoted by

$$\mathrm{cl}\,(E) \quad or \quad \overline{E}.$$

Observations

Let X be a metric space and let $A \subseteq X$. We have the following facts.

(i) $(\mathrm{int}\ A)^c = \mathrm{cl}\ (A^c)$,

(ii) $(\mathrm{cl}\ A)^c = \mathrm{int}\ (A^c)$.

Proof of (i). Observe that $\mathrm{int}\ A \subseteq A$ iff $A^c \subseteq (\mathrm{int}\ A)^c$. Then $\mathrm{cl}\ (A^c)$ as a closed set satisfies

$$\mathrm{cl}\ (A^c) \subseteq (\mathrm{int}\ A)^c.$$

Next we show that

$$\mathrm{cl}\ (A^c) \supseteq (\mathrm{int}\ A)^c.$$

Indeed, $(\mathrm{cl}\ (A^c))^c \subseteq A$ and $(\mathrm{cl}\ (A^c))^c$ is open, so $A^c \subseteq \mathrm{cl}\ (A^c)$, thus $(\mathrm{cl}\ (A^c))^c \subseteq \mathrm{int}\ (A)$, so

$$(\mathrm{int}\ A)^c \subseteq \mathrm{cl}\ (A^c).$$

This completes the proof. □

Examples of open and closed sets on $\mathbb{R}$

Let $-\infty \leq a < b \leq +\infty$.

Example 1

(a, b) is open in $\mathbb{R}$ (it is a ball).

Example 2

$\mathbb{Z}$ is closed in $\mathbb{R}$ since

$$\mathbb{Z}^c = \bigcup_{n \in \mathbb{Z}} (n, n+1)$$

is open.

Example 3

$[a, b]$ is closed in $\mathbb{R}$ since

$$[a, b]^c = (-\infty, a) \cup (b, +\infty) \quad \text{is open.}$$

Proposition

Proposition

Every open set in $\mathbb{R}$ is a countable disjoint union of open intervals.

Proof. If U is open, for each $x \in U$ consider the collection $\mathcal{F}_x$ of all open intervals I such that $x \in I \subseteq U$.

- It is easy to see that **the union of any family of open intervals containing a point in common is again an open interval** and hence

$$J_x = \bigcup_{I \in \mathcal{F}_x} I$$

  is an open interval.
- Moreover, it is the largest element of $\mathcal{F}_x$. If $x, y \in U$ then either

$$J_x = J_y \quad \text{or} \quad J_x \cap J_y = \emptyset.$$

  For otherwise $J_x \cup J_y$ would be a larger open interval than J_x in $\mathcal{F}_x$.

Proof

- Thus if
$$\mathcal{F} = \{\mathcal{F}_x \ : \ x \in U\}$$
then the distinct members of $\mathcal{F}$ are disjoint and
$$U = \bigcup_{J \in \mathcal{F}} J.$$

- For each $J \in \mathcal{F}$ pick a rational number $f(J) \in J$. The map $f : \mathcal{F} \to \mathbb{Q}$ is injective, for $J \neq J'$ then
$$J \cap J' = \emptyset \quad \text{and} \quad f(J) \neq f(J').$$
Hence $\mathrm{card}\,(\mathcal{F}) \leq \mathrm{card}\,(\mathbb{Q}) = \mathrm{card}\,(\mathbb{N})$, so $\mathcal{F}$ is countable. □

Dense sets

Dense set

Let (X, ρ) be a metric space, $E \subseteq X$ is said to be **dense** in X if

$$E \cap U \neq \emptyset$$

for every open set U in X.

- Equivalently, $E \subseteq X$ is said to be **dense** if $E \cap B(x, r) \neq \emptyset$ for every $x \in X$ and $r > 0$.

Examples

1. $\mathbb{Q}$ is dense in $\mathbb{R}$,
2. $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$,
3. $\Delta = \{k2^{-n} \ : \ 0 \leq k \leq 2^n\}$ is dense in $[0, 1]$,
4. if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\{n\alpha - \lfloor n\alpha \rfloor \ : \ n \in \mathbb{Z}\}$ is dense in $[0, 1]$.

Nowhere dense set

Nowhere dense set

Let (X, ρ) be a metric space, $E \subseteq X$ is said to be **nowhere dense** if the interior of the closure of E is empty, i.e.

$$\mathrm{int}\,(\mathrm{cl}\ E) = \emptyset.$$

Examples

1. $X = \mathbb{R}$, then $\{x\}$ for every $x \in \mathbb{R}$ is nowhere dense in $\mathbb{R}$.
2. $X = \mathbb{R}$, then $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

Separable space

Separable space

A metric space (X, ρ) is called **separable** if it has a countable dense subset.

Examples

1. $\mathbb{R}$ is separable since $\mathbb{Q}$ is dense and countable.
2. $\mathbb{R}^d$ is separable since $\mathbb{Q}^d$ is dense and countable.

Convergence in metric spaces

Convergence in metric spaces

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,ρ) is said **to converge** if there is a point $x \in X$ with the following property: For every $\varepsilon > 0$ there is an integer $N_\varepsilon \in \mathbb{N}$ such that

$$n \geq N_\varepsilon \quad \text{implies} \quad \rho(x_n, x) < \varepsilon.$$

- In this case we also say that $(x_n)_{n\in\mathbb{N}}$ **converges to** x or that x is **the limit** of $(x_n)_{n\in\mathbb{N}}$ and we write

$$x_n \xrightarrow[n\to\infty]{} x \quad \text{or} \quad \lim_{n\to\infty} x_n = x \quad \text{or} \quad \lim_{n\to\infty} \rho(x_n, x) = 0.$$

Divergence

If $(x_n)_{n=1}^{\infty}$ does not converge it is said **to diverge**.

Diameter of a set and bounded sets

Diameter of a set

In a metric space (X, ρ) we define **the diameter** of $E \subseteq X$ to be

$$\mathrm{diam}\,(E) = \sup\{\rho(x, y) \; : \; x, y \in E\}.$$

Bounded set

E is called **bounded** if $\mathrm{diam}\,(E) < \infty$.

Examples

1. $\{x\}$ is bounded, $\mathrm{diam}\,(\{x\}) = 0$,
2. $B(x, r) = \{y \in X \; : \; \rho(x, y) < r\}$ is bounded, since if $x_1, x_2 \in B(x, r)$, then $\rho(x_1, x_2) \leq \rho(x_1, x) + \rho(x, x_2) < 2r$. Thus $\mathrm{diam}\,(B(x, r)) \leq 2r$.
3. $(a, b) \subseteq \mathbb{R}$, $-\infty < a < b < \infty$, then $\mathrm{diam}\,((a, b)) = b - a$.

Bounded sequences

Bounded sequences

The sequence $(x_n)_{n\in\mathbb{N}}$ in (X, ρ) is said to be **bounded** if its range is bounded, i.e.

$$\mathrm{diam}\left(\{x_n \in X \ : \ n \in \mathbb{N}\}\right) < \infty.$$

Theorem

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, ρ).

(i) $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ iff every open set containing x contains x_n for all but finitely many $n \in \mathbb{N}$.

(ii) If $x, x' \in X$ and if $(x_n)_{n\in\mathbb{N}}$ converges to x and x', then $x = x'$.

(iii) If $(x_n)_{n\in\mathbb{N}}$ converges then $(x_n)_{n\in\mathbb{N}}$ is bounded.

Proof of (i)

Proof of (i) ($\Longrightarrow$). $\lim_{n\to\infty} \rho(x_n, x) = 0$ means that for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

(*)

$$n \geq N_\varepsilon \quad \text{implies} \quad \rho(x_n, x) < \varepsilon.$$

Take an open set V so that $x \in V$. Since V is open then there is $r > 0$ such that $B(x, r) \subseteq V$. It suffices to take $\varepsilon < r$ in (*) to see that $x_n \in B(x, r)$ for all $n \geq N_\varepsilon$ since $\rho(x_n, x) < \varepsilon$ by (*). □

Proof of (i) ($\Longleftarrow$). Conversely suppose that every open set V containing x contains all but finitely many of x_n's. Take $\varepsilon > 0$ and consider $V = B(x, \varepsilon)$, this set is open and $x \in V$. By assumption there exists $N \in \mathbb{N}$ (depending on V) such that $x_n \in B(x, \varepsilon)$ for all $n \geq N$. Thus $\rho(x_n, x) < \varepsilon$ if $n \geq N$. Hence $\lim_{n\to\infty} \rho(x_n, x) = 0$. □

Proof of (ii)

Proof of (ii). Let $\varepsilon > 0$ be given. There are $N_\varepsilon, N'_\varepsilon \in \mathbb{N}$ such that

$$n \geq N_\varepsilon \quad \text{implies} \quad \rho(x_n, x) < \frac{\varepsilon}{2},$$

$$n \geq N'_\varepsilon \quad \text{implies} \quad \rho(x_n, x') < \frac{\varepsilon}{2}.$$

Hence if $n \geq \max(N_\varepsilon, N'_\varepsilon)$, then by triangle inequality, we have

$$\rho(x, x') \leq \rho(x, x_n) + \rho(x', x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\rho(x, x') = 0$ and we are done. $\square$

Proof of (iii)

Proof of (iii). Suppose that $\lim_{n\to\infty} \rho(x_n, x) = 0$. Then there is $N \in \mathbb{N}$ so that

$$n \geq N \quad \text{implies} \quad \rho(x_n, x) < 1.$$

Let $r = \max\{1, \rho(x_1, x), \ldots, \rho(x_N, x)\}$. Then we see that

$$\rho(x_n, x) \leq r \quad \text{for all} \quad n \in \mathbb{N}.$$

Then

$$\mathrm{diam}(\{x_n \in X : n \in \mathbb{N}\}) \leq 2r,$$

since

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x, x_m) \leq 2r \quad \text{for any} \quad m, n \in \mathbb{N}.$$

This completes the proof. □

Proposition

Proposition

If (X, ρ) is a metric space, $E \subseteq X$ and $x \in X$, the following are equivalent.

- (a) $x \in \mathrm{cl}\,(E)$,
- (b) $B(x, r) \cap E \neq \emptyset$ for all $r > 0$,
- (c) There is a sequence $(x_n)_{n\in\mathbb{N}}$ in E that converges to x.

Proof $(a) \implies (b)$**.** If $B(x, r) \cap E = \emptyset$, then $B(x, r)^c$ is a closed set containing E but not x, so

$$E \subseteq \mathrm{cl}\,(E) \subseteq B(x, r)^c$$

and consequently $x \notin \mathrm{cl}\,(E)$. □

Proof $(b) \implies (a)$**.** If $x \notin \mathrm{cl}\,(E)$, since $(\mathrm{cl}\,(E))^c$ is open there is $r > 0$ so that $B(x, r) \subseteq (\mathrm{cl}\,(E))^c \subseteq E^c$, so $B(x, r) \cap E = \emptyset$. □

Proof

Proof $(b) \implies (c)$**.** For each $n \in \mathbb{N}$ there exists $x_n \in B(x, 1/n) \cap E$ hence

$$\rho(x_n, x) < \frac{1}{n}$$

and consequently

$$\lim_{n\to\infty} \rho(x_n, x) = 0$$

as desired. □

Proof $(c) \implies (b)$**.** If $B(x, r) \cap E = \emptyset$, then

$$\rho(y, x) \geq r$$

for all $y \in E$, so no sequence of E can converge to x. □

Accumulation and isolated points

Accumulation point

Let (X, ρ) be a metric space, $x \in X$ is called **an accumulation point of** $E \subseteq X$ if for every open set $U \ni x$ we have

$$(E \setminus \{x\}) \cap U \neq \emptyset.$$

An accumulation point x of $E \subseteq X$ is sometimes also called **a limit point of** E or **a cluster point of** E.

Isolated point

A point $x \in E$ is called **an isolated point of** E if it is not an accumulation point of E.

Examples

Example 1

$X = \mathbb{R}$, $E = [a, b]$, $-\infty < a < b < \infty$, each $x \in E$ is an accumulation point of E.

Example 2

$X = \mathbb{R}$, $E = (a, b)$, $-\infty < a < b < \infty$, each point of $x \in [a, b]$ is an accumulation point of E.

Example 3

$X = \mathbb{R}$, $E = \mathbb{Z}$, each point of $\mathbb{Z}$ is an isolated point.

Remark

If $x \in X$ is a limit point of $E \subseteq X$ it does not need to be an element of E.

Perfect space

Perfect space

A set $E \subseteq X$ of a metric space is called **perfect** if $E = \mathrm{acc}\,(E)$, where $\mathrm{acc}\,(E)$ is the set of all accumulation points of E.

Proposition

Let (X, ρ) be a metric space, let $E \subseteq X$, then $\mathrm{cl}\,(E) = E \cup \mathrm{acc}\,(E)$ and E is closed if $\mathrm{acc}\,(E) \subseteq E$.

Proof. **($\Longrightarrow$)** If $x \notin \mathrm{cl}\,(E)$ then there is $B(x, \varepsilon) \subseteq (\mathrm{cl}\,(E))^c$, thus $B(x, \varepsilon) \cap E = \emptyset$ hence $x \notin \mathrm{acc}\,(E)$. Thus $E \cup \mathrm{acc}\,(E) \subseteq \mathrm{cl}\,(E)$.

($\Longleftarrow$) If $x \notin E \cup \mathrm{acc}\,(E)$ there is an open $U \ni x$ such that $U \cap E = \emptyset$. Then $\mathrm{cl}\,(E) \subseteq U^c$ so $x \notin \mathrm{cl}\,(E)$ thus $\mathrm{cl}\,(E) \subseteq E \cap \mathrm{acc}\,(E)$.

Finally, E is closed iff $E = \mathrm{cl}\,(E)$ iff $\mathrm{acc}\,(E) \subseteq E$, since

$$E = \mathrm{cl}\,(E) = E \cup \mathrm{acc}\,(E) \quad \text{as desired.} \qquad \square$$

Boundary

Boundary of E

The difference $\text{cl}(E) \setminus \text{int}(E)$ is called **the boundary of E** in a metric space (X, ρ) and it is denoted by ∂E.

Example 1

$X = \mathbb{R}$ and $-\infty < a < b < \infty$, $E = (a, b)$, then $\text{cl}(E) = [a, b]$, $\text{int}(E) = (a, b)$, so $\partial E = \{a, b\}$.

Example 2

$X = \mathbb{R}$, $E = \mathbb{Z}$, then $\text{int}(E) = \emptyset$, $\text{cl}(\mathbb{Z}) = \mathbb{Z}$, thus $\partial \mathbb{Z} = \mathbb{Z}$.

Example 3

$X = \mathbb{R}$, $E = \mathbb{Q}$, $\text{cl}(E) = \mathbb{R}$, $\text{int}(E) = \emptyset$, so $\partial \mathbb{Q} = \mathbb{R}$.

Boundary - examples

Example 4

$X = \mathbb{R}^2$, $\rho(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$, $x = (x_1, x_2)$, $y = (y_1, y_2)$,

$$E = B(x,r) = \{y \in \mathbb{R}^2 \ : \ \rho(x,y) < r\},$$

$$\mathrm{cl}\,(E) = \{y \in \mathbb{R}^2 \ : \ \rho(x,y) \leq r\},$$

$$\partial E = \{y \in \mathbb{R}^2 \ : \ \rho(x,y) = r\}.$$

Boundary - example