Lecture 15
Complete spaces and Compact sets

MATH 411H, FALL 2025

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Proposition

Proposition

Let (X, ρ) be a metric space, $E \subseteq X$ is closed iff for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$ such that $x_n \xrightarrow[n \to \infty]{} x \in X$ we have $x \in E$.

Proof. ($\Longrightarrow$) Suppose that E is closed and consider $(x_n)_{n \in \mathbb{N}} \subseteq E$ such that $x_n \xrightarrow[n \to \infty]{} x \in X$. We have to show that $x \in E$. Observe that

$$B(x, r) \cap E \neq \emptyset \quad \text{for any} \quad r > 0.$$

But $x_n \xrightarrow[n \to \infty]{} x$ iff $x_n \in B(x, r)$ for all but finitely many $n \in \mathbb{N}$, and consequently we conclude that $x \in \mathrm{cl}\,(E)$, hence $x \in E$.

($\Longleftarrow$) Conversely, if $x \in \mathrm{cl}\,(E)$ then there is a sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$ so that $x_n \xrightarrow[n \to \infty]{} x \in \mathrm{cl}\,(E) \subseteq X$ thus by our assumption $x \in E$. $\qquad \square$

Theorem

Theorem

The subsequential limits of a sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, ρ) form a closed subset of X.

Proof. Let E^* be the set of subsequential limits of $(x_n)_{n \in \mathbb{N}}$ and let q be an accumulation point of E^*. We will show that $q \in E^*$.

- Choose $n_1 \in \mathbb{N}$ so that $E^* \ni x_{n_1} \neq q$ (if no such point exists then E^* has only one point and there is nothing to prove). Set

$$\delta = \rho(x_{n_1}, q) > 0.$$

- Suppose that $n_1, \ldots, n_{i-1}$ have been chosen. Since q is an accumulation point of E^* there is $x \in E^*$ so that

$$\rho(x, q) < \delta 2^{-i-1}.$$

Since $x \in E^*$ there is $n_i > n_{i-1}$ such that $\rho(x, x_{n_i}) < \delta 2^{-i-1}$.

Proof

- Hence, by the triangle inequality

$$\rho(q, x_{n_i}) \leq \rho(q, x) + \rho(x, x_{n_i}) < \delta 2^{-i-1} + \delta 2^{-i-1} = \delta 2^{-i}.$$

- This means that $(x_{n_i})_{i \in \mathbb{N}}$ converges to q, i.e.

$$\lim_{i \to \infty} x_{n_i} = q \quad \Longleftrightarrow \quad \lim_{i \to \infty} \rho(x_{n_i}, q) = 0$$

  thus $q \in E^*$.

- In fact, we have shown that

$$\mathrm{acc}\,(E^*) \subseteq E^*,$$

  which means that E^* is closed. $\qquad \qquad \Box$

Cauchy sequences

Cauchy sequence

A sequence $(x_n)_{n \in \mathbb{N}}$ is a metric space (X, ρ) is said to be **a Cauchy sequence** if for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

$$m, n \geq N_\varepsilon \quad \text{implies} \quad \rho(x_m, x_n) < \varepsilon.$$

Complete spaces

A subset of a metric space (X, ρ) is called **complete** if every Cauchy sequence in E converges and its limit is in E.

Complete spaces - examples

Example 1

The set of real numbers $\mathbb{R}$ is complete.

Example 2

The open unit interval $(0, 1)$ is not complete space in $\mathbb{R}$.

- Indeed, let $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$, then $x_n \in (0, 1)$ and $(x_n)_{n \in \mathbb{N}}$ is Cauchy in $(0, 1)$, but 0, which is the limit of $(x_n)_{n \in \mathbb{N}}$ is not in $(0, 1)$.

Example 3

$[0, 1]$ is complete space in $\mathbb{R}$.

Some facts

Fact 1

If $(x_n)_{n\in\mathbb{N}}$ is Cauchy in a metric space (X, ρ) then $(x_n)_{n\in\mathbb{N}}$ is bounded.

Fact 2

If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a metric space (X, ρ) and $\lim_{k\to\infty} \rho(x_{n_k}, x) = 0$ for some $(x_{n_k})_{k\in\mathbb{N}}$, then

$$\lim_{n\to\infty} \rho(x_n, x) = 0.$$

Proposition

A closed subset of a complete metric space is complete and a complete subset of an arbitrary metric space is closed.

Proof of the Proposition

Proof. If (X, ρ) is complete, $E \subseteq X$ is closed and $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in E, then $(x_n)_{n \in \mathbb{N}}$ has a limit in X. But $\mathrm{cl}(E) = E$, thus $x \in \mathrm{cl}(E)$, so $x \in E$.

If $E \subseteq X$ is complete and $x \in \mathrm{cl}(E)$ then we know that there exists $(x_n)_{n \in \mathbb{N}} \subseteq E$ converging to x. But $(x_n)_{n \in \mathbb{N}}$ is Cauchy so its limit lies in E, thus $\mathrm{cl}(E) = E$ as desired. □

Remark

In the second part of the proof we have used the fact that if $(x_n)_{n \in \mathbb{N}}$ converges (say to x in a metric space (X, ρ)) then $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

Cantor intersection theorem

Cantor intersection theorem

A metric space is complete iff for every decreasing sequence

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \dots$$

of nonempty closed sets in X with $\mathrm{diam}\,(F_n) \xrightarrow[n \to \infty]{} 0$, one has

$$\bigcap_{n \in \mathbb{N}} F_n = \{x_0\} \quad \text{for some} \quad x_0 \in X.$$

Proof ($\Longrightarrow$):

Assume that (X, ρ) is complete.

- Let $(F_n)_{n \in \mathbb{N}}$ be such that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ and $\text{diam}(F_n) \xrightarrow[n \to \infty]{} 0$.
- Choose $x_n \in F_n$, let $\varepsilon > 0$ and pick $N_\varepsilon \in \mathbb{N}$ such that $\text{diam}(F_n) < \varepsilon$ for all $n \geq N_\varepsilon$. Note that for $n \geq m \geq N_\varepsilon$ we have

$$x_n \in F_n \subseteq F_m,$$

so

$$\rho(x_n, x_m) \leq \text{diam}(F_m) < \varepsilon.$$

- This ensures that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence and, consequently, converges to some $x_0 \in X$. Since each F_n is closed then $x_0 \in F_n$ for all $n \in \mathbb{N}$, thus $x_0 \in \bigcap_{n \in \mathbb{N}} F_n$.
- Suppose there is $y \neq x_0$ so that $y \in \bigcap_{n \in \mathbb{N}} F_n$, then

$$0 < \rho(x_0, y) \leq \text{diam}(F_n) \xrightarrow[n \to \infty]{} 0,$$

contradiction. Thus $\bigcap_{n \in \mathbb{N}} F_n = \{x_0\}$.

Proof ($\Longleftarrow$):

To prove the converse implication assume that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

- Let

$$F_n = \mathrm{cl}\left(\{x_m \ : \ m \geq n\}\right).$$

Fact

$$\mathrm{diam}\,(E) = \mathrm{diam}\,(\mathrm{cl}\,(E)).$$

- We see that $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ and

$$\mathrm{diam}\,(F_n) = \mathrm{diam}\,(\{x_m \ : \ m \geq n\}) \xrightarrow[n\to\infty]{} 0.$$

Thus $\bigcap_{n\in\mathbb{N}} F_n = \{x_0\}$ for some $x_0 \in X$.

- Finally, we conclude $\lim_{n\to\infty} \rho(x_n, x_0) = 0$ as desired. $\qquad\square$

Coverings and compact sets

Coverings

Let (X, ρ) be a metric space.

- If $E \subseteq X$ and $(V_\alpha)_{\alpha \in A}$ is a family of sets such that $E \subseteq \bigcup_{\alpha \in A} V_\alpha$, then $(V_\alpha)_{\alpha \in A}$ is called **a cover** of E and E is said to be **covered** by the V_α's.

- If additionally each V_α is open $(V_\alpha)_{\alpha \in A}$ is called **an open cover of E.**

Heine–Borel property

A subset K of a metric space (X, ρ) is said to be **compact** if every open cover of K contains a finite subcover. More explicitly, if $(V_\alpha)_{\alpha \in A}$ is an open cover of K then there are finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}.$$

Compact sets - examples

Example 1

Every finite subset of $\mathbb{R}$ is compact.

Example 2

$K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$.

- Indeed, let $(V_\alpha)_{\alpha \in A}$ be an open cover of K, then there is $\alpha_0 \in A$ such that $0 \in V_{\alpha_0}$ since $\lim_{n \to \infty} \frac{1}{n} = 0$ and V_{α_0} is open thus it contains all but finitely many $\frac{1}{n}$'s. In other words, there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $\frac{1}{n} \in V_{\alpha_0}$. Then, for each $j \in \{1, 2, \ldots, n_0 - 1\}$ we can pick $\alpha_j \in A$ so that $\frac{1}{j} \in V_{\alpha_j}$ and we see

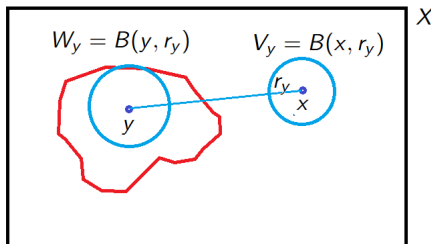$$K \subseteq \bigcup_{j=0}^{n_0} V_{\alpha_j}.$$

Theorem

Theorem

Compact subsets of metric spaces are closed.

Proof. Let K be compact subset of a metric space X.

- We shall prove that K^c is open in X. Let $x \in X \setminus K$. If $y \in K$, let

$$V_y = B(x, r_y) \quad \text{and} \quad W_y = B(y, r_y),$$

where $r_y < \frac{1}{2}\rho(x, y)$, then $V_y \cap W_y = \emptyset$.

Proof

- Since K is compact $K \subseteq \bigcup_{y \in K} W_y$, then we can find $y_1, \ldots, y_n \in K$ so that

$$K \subseteq \bigcup_{j=1}^{n} W_{y_j} = W.$$

- If $V = V_{y_1} \cap \ldots \cap V_{y_n}$ then V is an open set containing x and

$$V \cap W = \emptyset.$$

- Hence $x \in V \subseteq W^c \subseteq K^c$ thus x is an interior point of K^c. $\qquad\square$

Theorem

Theorem

Closed subsets of compact sets are compact.

Proof. Suppose that $F \subseteq K \subseteq X$ and F is closed in X and K is compact.

- Let $(V_\alpha)_{\alpha \in A}$ be an open cover of F. Observe that

$$F \subseteq \underbrace{K \subseteq \bigcup_{\alpha \in A} V_\alpha}_{F \subseteq} \cup \underbrace{F^c}_{\text{open}}.$$

- The set K is compact thus there is a finite subcover of

$$(V_\alpha)_{\alpha \in A} \cup \{F^c\}$$

that covers K.

- But $F \subseteq K$ hence this is also a finite subcover of F upon removing F^c as desired. $\quad\square$

Theorem

Theorem

If $(K_\alpha)_{\alpha \in A}$ is a collection of compact sets of a metric space (X, ρ) such that the intersection of every finite subcollection of $(K_\alpha)_{\alpha \in A}$ is non-empty then

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

Proof. Fix a member K_{α_0} of $(K_\alpha)_{\alpha \in A}$ and set $G_\alpha = K_\alpha^c$.

- Suppose that

$$\bigcap_{\alpha \in A} K_\alpha = K_{\alpha_0} \cap \left(\bigcap_{\alpha \in A \setminus \{\alpha_0\}} K_\alpha \right) = \emptyset.$$

Proof

- Then

$$K_{\alpha_0} \subseteq \bigcup_{\alpha \in A \setminus \{\alpha_0\}} G_\alpha.$$

- Since K_{α_0} is compact there are $\alpha_1, \ldots, \alpha_n \in A$ so that

$$K_{\alpha_0} \subseteq \bigcup_{j=1}^{n} G_{\alpha_j}.$$

- Hence

$$K_{\alpha_0} \cap K_{\alpha_1} \cap \ldots \cap K_{\alpha_n} = \emptyset,$$

which is a contradiction. So we must have

$$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

as desired. □

Totally bounded sets

Totally bounded set

Let (X, ρ) be a metric space, $E \subseteq X$ is called **totally bounded** if for every $\varepsilon > 0$, the set E can be covered by finitely many balls of radius ε.

- It means that there is $N_\varepsilon \in \mathbb{N}$ so that

$$E \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \quad \text{for some} \quad x_1, x_2, \ldots, x_{N_\varepsilon} \in X.$$

Remark 1

If E is totally bounded so is $\mathrm{cl}\,(E)$. Indeed,

$$E \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon) \implies \mathrm{cl}\,(E) \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, 2\varepsilon).$$

Remark

Remark 2

Every totally bounded set E is bounded. If

$$x, y \in E \subseteq \bigcup_{j=1}^{N_\varepsilon} B(x_j, \varepsilon),$$

then say $x \in B(x_1, \varepsilon)$, $y \in B(x_2, \varepsilon)$ and

$$\rho(x, y) \leq \rho(x, x_1) + \rho(x_1, x_2) + \rho(x_2, y)$$
$$\leq \varepsilon + \max\{\rho(x_i, x_j) \,:\, 1 \leq i, j \leq N_\varepsilon\} + \varepsilon.$$

- The converse is false in general.

Characterization of compactness

Theorem

If E is a subset of a metric space (X, ρ) the following are equivalent.

(a) E is complete and totally bounded.

(b) (The Bolzano–Weierstrass property) Every sequence in E has a subsequence that converges to a point of E.

(c) (The Heine–Borel property) If $(V_\alpha)_{\alpha \in A}$ is an open cover of E then there is finite $F \subseteq A$ such that $(V_\alpha)_{\alpha \in F}$ covers E.

Remark

This theorem can be thought of as a characterization of compactness in metric spaces.

Proof of (a) $\Rightarrow$ (b): 1/2

Suppose that (a) holds and $(x_n)_{n \in \mathbb{N}} \subseteq E$. We find $(x_{n_k})_{k \in \mathbb{N}}$ such that $\rho(x_{n_k}, x_0) \xrightarrow[k \to \infty]{} 0$ for some $x_0 \in E$.

- E can be covered by finitely many balls of radius $1/2$. At least one of them must contain x_n for infinitely many $n \in \mathbb{N}$:
    - say $x_n \in B_1$ for $n \in \mathbb{N}_1 \subseteq \mathbb{N}$ and $\mathrm{card}\,(\mathbb{N}_1) = \mathrm{card}\,(\mathbb{N})$.

- Now $E \cap B_1$ can be covered by finitely many balls of radius $1/4$. At least one of them must contain x_n for infinitely many $n \in \mathbb{N}$:
    - say $x_n \in B_2$ for $n \in \mathbb{N}_2 \subseteq \mathbb{N}_1$ and $\mathrm{card}\,(\mathbb{N}_2) = \mathrm{card}\,(\mathbb{N})$.

- Continuing inductively we obtain a sequence of balls B_j of radius 2^{-j} and decreasing sequence of subsets $\mathbb{N}_j$ of $\mathbb{N}$ such that
    - $x_n \in B_j$ for $n \in \mathbb{N}_j$, $\mathbb{N}_{j+1} \subseteq \mathbb{N}_j \subseteq \mathbb{N}$, $\mathrm{card}\,(\mathbb{N}_j) = \mathrm{card}\,(\mathbb{N})$.

Proof of (a) $\Rightarrow$ (b): 2/2

- Pick $n_1 \in \mathbb{N}_1$, $n_2 \in \mathbb{N}_2, \ldots$ such that

$$n_1 < n_2 < n_3 < \ldots.$$

- Then $(x_{n_j})_{j \in \mathbb{N}}$ is a Cauchy sequence for

$$\rho(x_{n_j}, x_{n_k}) < 2^{1-j} \quad \text{if} \quad k \geq j,$$

since $x_{n_j}, x_{n_k} \in B_j$ and

$$\operatorname{diam}(B_j) \leq 2^{1-j}.$$

- Since E is complete the sequence $(x_{n_k})_{k \in \mathbb{N}}$ has a limit in E and the implication (a) $\Rightarrow$ (b) is proved. $\qquad\square$

Proof of (b) $\Rightarrow$ (a)

We show that of either condition in (a) fails then so does (b).

- If E is not complete there is a Cauchy sequence $(x_n)_{n \in \mathbb{N}} \subseteq E$, with no limit in E. No subsequence of $(x_n)_{n \in \mathbb{N}}$ can converge in E, for otherwise the whole sequence would converge to the same limit.

- On the other hand if E is not totally bounded, let $\varepsilon > 0$ be such that E cannot be covered by finitely many balls of radius $\varepsilon > 0$. Choose $x_n \in E$ inductively as follows. Let $x_1 \in E$, and having chosen $x_1, \ldots, x_n$ pick

$$x_{n+1} \in E \setminus \bigcup_{j=1}^{n} B(x_j, \varepsilon),$$

then $\rho(x_n, x_m) \geq \varepsilon$ for all $n \neq m$, so $(x_n)_{n \in \mathbb{N}}$ has no convergent subsequence. Thus (b) $\Rightarrow$ (a). $\qquad \square$

Proof of theorem (a) and (b) $\Rightarrow$ (c)

It suffices to show that if (b) holds and $(V_\alpha)_{\alpha \in A}$ is an open cover of E then the following claim holds:

Claim

There exists $\varepsilon > 0$ such that every ball of radius $\varepsilon > 0$ that intersects E is contained in some V_α.

Then E can be covered by finitely many such balls by (a) this allows us to find a finite subcover of $(V_\alpha)_{\alpha \in A}$.

Proof of Claim

Suppose for a contradiction that the claim is not true.

- For each $n \in \mathbb{N}$ there is a ball B_n of radius 2^{-n} such that $B_n \cap E \neq \emptyset$ and B_n is contained in no V_α.
- Pick $x_n \in B_n \cap E$. Using (b), (by passing to a subsequence if necessary) we may assume $\lim_{n \to \infty} \rho(x_n, x) = 0$ for some $x \in E$.
- We have $x \in V_\alpha$ for some $\alpha \in A$ and since V_α is open there is $\varepsilon > 0$ so that $B(x, \varepsilon) \subseteq V_\alpha$.
- If n is large enough so that $\rho(x_n, x) < \frac{\varepsilon}{3}$ and $2^{-n} < \frac{\varepsilon}{3}$, then $B_n \subseteq B(x, \varepsilon) \subseteq V_\alpha$, which is contradiction.
- Indeed, pick $y \in B_n$, then

$$\rho(y, x) \leq \rho(x_n, y) + \rho(x_n, x) < 2^{1-n} + \frac{\varepsilon}{3} \leq \varepsilon.$$

This completes the proof of the implication (a) and (b) $\Rightarrow$ (c). $\qquad\square$

Proof of (c) $\Rightarrow$ (b)

- If $(x_n)_{n \in \mathbb{N}} \subseteq E$, with no convergent sequence, for each $x \in E$ there is a ball B_x centered at x that contains x_n for only finitely many n.
- Otherwise, some sequence would converge to x. Then

$$(B_x)_{x \in E}$$

is a cover of E by open sets with no finite subcover. $\qquad\square$