Lecture 16

Compact sets, Perfect Sets, Connected Sets, and Cantor set

MATH 411H, FALL 2025

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Compactness in Euclidean spaces

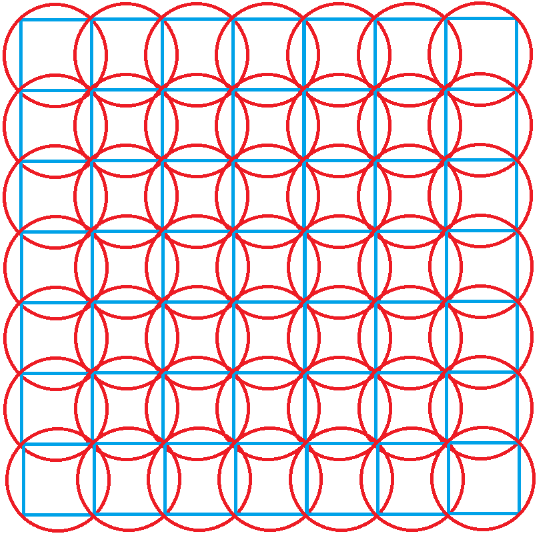
Theorem

Every closed and bounded set of $\mathbb{R}^n$ is complete.

Proof. We deduce compactness by showing completeness and total boundedness.

- Since every closed subset of $\mathbb{R}^n$ is complete is suffices to show that bounded subsets of $\mathbb{R}^n$ are totally bounded.
- Since every bounded set is contained in some cube $Q = [-R, R]^n$ it is enough to show that Q is totally bounded.
- Given $\varepsilon > 0$ pick the integer $k > \frac{R\sqrt{n}}{\varepsilon}$ and express Q as the union of n^n congruent subcubes by dividing the interval $[-R, R]$ into k equal pieces.
- The side length of these subcubes is $\frac{2R}{k}$ and hence the diameter is $\sqrt{n}\left(\frac{2R}{k}\right) < 2\varepsilon$, so they are contained in the balls of radius ε about their centers. □

$Q = [-R, R]^n$ is totally bounded



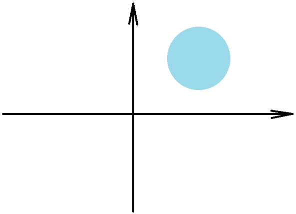
Example

Example

Determine if the set

$$X = \{(x, y) \in \mathbb{R}^2 \; : \; (x-1)^2 + (y-1)^2 < 1\}$$

is compact or not in $\mathbb{R}^2$ with Euclidean metric.



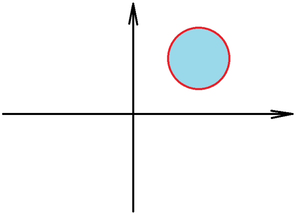
Solution. Note that $(2, 0)$ is an accumulation point of X, but $(2, 0) \notin X$. Therefore, X **is not closed**, so it is **not compact**. □

Example

Example

Determine if the set is compact or not in $\mathbb{R}^2$ with Euclidean metric:

$$X = \{(x, y) \in \mathbb{R}^2 \ : \ (x-1)^2 + (y-1)^2 \leq 1\}.$$



Solution. X contains all of its accumulation points so it is **closed**. It is contained in the ball $B(0, 10)$, so it is **bounded**. Therefore, by the previous theorem, it is **compact**. □

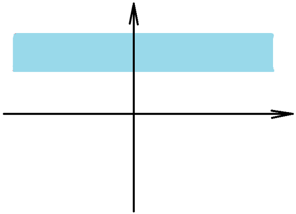
Example

Example

Determine if the set

$$X = \{(x, y) \in \mathbb{R}^2 \; : 1 < y < 2\}$$

is compact or not in $\mathbb{R}^2$ with Euclidean metric.



Solution. Note that $(0, 2)$ is an accumulation point of X, but $(0, 2) \notin X$. Therefore, X **is not closed**, so it is **not compact**. □

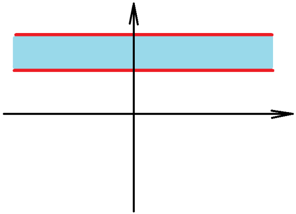
Example

Example

Determine if the set

$$X = \{(x, y) \in \mathbb{R}^2 \; : 1 \leq y \leq 2\}$$

is compact or not in $\mathbb{R}^2$ with Euclidean metric.



Solution. In can be checked that X is closed, although it is not contained in any ball, so it is **not bounded**, so it is **not compact**. □

Examples

Example

Determine if the set $\mathbb{Q}$ is compact in $\mathbb{R}$.

Solution. $\mathbb{Q}$ is not contained in any interval, so it is **not compact**. □

Example

Determine if the set $\mathbb{Q} \cap [0,1]$ is compact in $\mathbb{R}$.

Solution. $\mathbb{Q}$ is contained in $(-1,2)$, but $\mathrm{cl}\,\mathbb{Q} \cap [0,1] = [0,1] \neq \mathbb{Q} \cap [0,1]$, so it is not closed, so it is **not compact**. □

Accumulation and isolated points

Accumulation point

Let (X, ρ) be a metric space, $x \in X$ is called **an accumulation point of** $E \subseteq X$ if for every open set $U \ni x$ we have

$$(E \setminus \{x\}) \cap U \neq \emptyset.$$

An accumulation point x of $E \subseteq X$ is sometimes also called **a limit point of** E or **a cluster point of** E.

Isolated point

A point $x \in E$ is called **an isolated point of** E if it is not an accumulation point of E.

Perfect sets

Perfect sets

We say that a subset E of a metric space (X, ρ) is **perfect** if E is closed and every point of E is its limit point or equivalently

$$E = \mathrm{acc}\, E.$$

Theorem

Let $\emptyset \neq P \subseteq \mathbb{R}^k$ be a perfect set. Then P is uncountable.

In the proof we will use the fact that we have just proved:

Proposition

Every closed and bounded set of $\mathbb{R}^k$ is compact.

Proof: 1/3

Proof. Since P has limit points, P must be infinite. In fact, for every $x \in P$ and $r > 0$

$$B(x, r) \cap P \quad \text{is infinite.}$$

- Suppose not, i.e. there is $x_0 \in P$ and $r_0 > 0$ such that

$$B(x_0, r_0) \cap P = \{x_1, \ldots, x_n\}.$$

- Consider

$$\rho(x_0, x_1), \ldots, \rho(x_0, x_n)$$

  and let

$$r = \min_{1 \leq i \leq n} \rho(x_0, x_i) > 0.$$

- Then

$$B(x_0, r) \cap P = \emptyset,$$

  thus x_0 is not a limit point, contradiction.

Proof: 2/3

Now we can assume $\mathrm{card}\,(P) \geq \mathrm{card}\,(\mathbb{N})$. Suppose for a contradiction that $\mathrm{card}\,(P) = \mathrm{card}\,(\mathbb{N})$, i.e. $P = \{x_1, x_2, \ldots\}$.

- Let $V_1 = B(x_1, r)$, then of course $V_1 \cap P \neq \emptyset$. Suppose that V_n has been constructed so that $V_n \cap P \neq \emptyset$.
- Since every point of P is a limit point of P there is an open set V_{n+1} such that
  1. (i) $\mathrm{cl}\,(V_{n+1}) \subseteq V_n$,
  2. (ii) $x_n \notin \mathrm{cl}\,(V_{n+1})$,
  3. (iii) $V_{n+1} \cap P \neq \emptyset$.
- Let $K_n = \mathrm{cl}\,(V_n) \cap P$, this set is closed and bounded, thus compact. Since $x_n \notin K_{n+1}$, no point of P lies in $\bigcap_{n=1}^{\infty} K_n$, but $K_n \subseteq P$, so

$$\bigcap_{n=1}^{\infty} K_n = \emptyset.$$

Proof: 3/3

- On the other hand, $K_n \neq \emptyset$, compact, and $K_{n+1} \subseteq K_n$, and the family K_n has a finite intesection property, i.e. any finite intersection of members of $(K_n)_{n\in\mathbb{N}}$ is nonempty,

$$K_{n_1} \cap \ldots \cap K_{n_k} \neq \emptyset.$$

- Thus

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset,$$

which is a contradiction. Hence P must be uncountable. □

Corollary

Every interval $[a, b]$ with $a < b$, and also $\mathbb{R}$ are uncountable.

Separated and connected sets

Separated sets

Two subsets A and B of a metric space (X, ρ) are said to be **separated** if both

$$A \cap \mathrm{cl}\,(B) = \emptyset \quad \text{and} \quad \mathrm{cl}\,(A) \cap B = \emptyset.$$

In other words, no points of A lies in the closure of B and vice versa.

Connected set

A set $E \subseteq X$ is said to be **connected** if E is not a union of two nonempty separated sets.

Example

- $[0, 1]$ and $(1, 2)$ are not separated since 1 is a limit point of $(1, 2)$.
- However, $(0, 1)$ and $(1, 2)$ are separated.

Theorem

Theorem

$E \subseteq \mathbb{R}$ is connected iff for all $x, y \in E$ if $x < z < y$, then $z \in E$.

Proof ($\Longrightarrow$). If there exist $x, y \in E$ and $z \in (x, y)$ such that $z \notin E$, then

$$E = A_z \cup B_z, \quad \text{where} \quad A_z = E \cap (-\infty, z) \quad \text{and} \quad B_z = E \cap (z, \infty).$$

Since $x \in A_z$ and $y \in B_z$, then $A_z \neq \emptyset$, $B_z \neq \emptyset$ and also $A_z \subseteq (-\infty, z)$, $B_z \subseteq (z, \infty)$, so they are separated. Hence E is not connected.

Proof

Proof ($\Longleftarrow$). Conversely, suppose that E is not connected.

- Then there are non-empty separated sets A, B such that $A \cup B = E$.
- Pick $x \in A$ and $y \in B$ and without loss of generality assume $x < y$. Define

$$z = \sup\left(A \cap [x, y]\right).$$

hence $z \in \mathrm{cl}\,(A)$ and $z \notin B$. In particular, $x \leq z < y$.

- If $z \notin A$ it follows $x < z < y$ and $z \notin E$.
- If $z \in A$ then $z \notin \mathrm{cl}\,(B)$ hence there is z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$. □

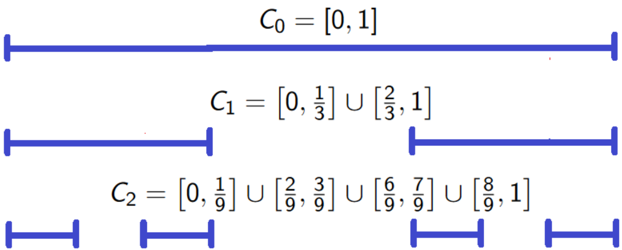
Example

Prove that $X = \mathbb{R} \setminus \{0\}$ is not connected.

Solution. We have $-1, 1 \in X$, but $-1 < 0 < 1$ and $0 \notin X$, so X is not connected. □

There exists a perfect set in $\mathbb{R}$ which contains no segment.

- Let $C_0 = [0, 1]$. Given C_n that consist of 2^n disjoint closed intervals each of length 3^{-n} take each of these intervals and delete the open middle third to produce two closed intervals each of length 3^{-n-1}.



- Take C_{n+1} to be the union of 2^{n+1} closed intervals so formed and continue.

Cantor set

Cantor set

The set

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$$

is called **the Cantor set** or ternary Cantor set.

- Each $C_0 \supseteq C_1 \supseteq C_2 \supseteq \dots$ is closed and bounded thus compact, and the family $(C_n)_{n\in\mathbb{N}}$ has finite intersection property thus the Cantor set is compact and $\mathcal{C} \neq \emptyset$.

Property (*)

By the construction for each $k, m \in \mathbb{N}$ we see that no segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \qquad \text{has a point in common with } \mathcal{C}.$$

Properties of the Cantor set

- Since every segment (α, β) contains a segment of the form (*) if m is sufficiently large, since the set

$$\left\{ \frac{\ell}{3^m} \; : \; m \in \mathbb{N} \text{ and } 0 \leq \ell \leq 3^m - 1 \right\}$$

is dense in $[0,1]$. Thus $\mathcal{C}$ contains no segment (α, β). This also shows $\operatorname{int} \mathcal{C} = \emptyset$.

- To prove that $\mathcal{C}$ is perfect it is enough to show that $\mathcal{C}$ contains no isolated point. Let $x \in \mathcal{C}$ and let I_n be the unique interval from C_n which contains $x \in I_n$. Let x_n be the endpoint of I_n such that $x \neq x_n$. It follows from the construction of $\mathcal{C}$ that $x_n \in \mathcal{C}$. Hence x is a limit point of $\mathcal{C}$ thus $\mathcal{C}$ is perfect.

More about Cantor set

- Each component of C_n can be described as the set

$$C_n = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_j}{3^j} \ : \ \varepsilon_j \in \{0, 1, 2\} \text{ and } \varepsilon_j \neq 1 \text{ for } 1 \leq j \leq n \right\}.$$

- Consequently,

$$\mathcal{C} = \left\{ \sum_{n=1}^{\infty} \frac{\varepsilon_j}{3^j} \ : \ \varepsilon_j \in \{0, 2\} \right\}.$$

Fact

Fact

Any number $\sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j}$ is uniquely determined by its sequence $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$ with $\varepsilon_j \in \{0, 2\}$.

Proof. Take $\varepsilon = (\varepsilon_j)_{j\in\mathbb{N}}$, $\delta = (\delta_j)_{j\in\mathbb{N}}$ with $\varepsilon_j, \delta_j \in \{0, 2\}$ such that $\varepsilon \neq \delta$. Let $N = \min\{j \in \mathbb{N} : \varepsilon_j \neq \delta_j\}$ and assume $0 = \varepsilon_N < \delta_N = 2$. Then

$$\sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = \sum_{j=1}^{N-1} \frac{\varepsilon_j}{3^j} + \sum_{j=N+1}^{\infty} \frac{\varepsilon_j}{3^j} \leq \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^{N+1}} \sum_{j=0}^{\infty} \frac{1}{3^j}$$

$$\leq \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^{N+1}} \underbrace{\frac{1}{1-\frac{1}{3}}}_{\frac{3}{2}} = \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{1}{3^N} < \sum_{j=1}^{N-1} \frac{\delta_j}{3^j} + \frac{2}{3^N} \leq \sum_{j=1}^{\infty} \frac{\delta_j}{3^j}.$$

This completes the proof. □

Remarks

Remark

We have two different representations

$$\frac{1}{3} = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = A, \quad \varepsilon_1 = 1, \quad \varepsilon_j = 0 \quad \text{for} \quad j \geq 2.$$

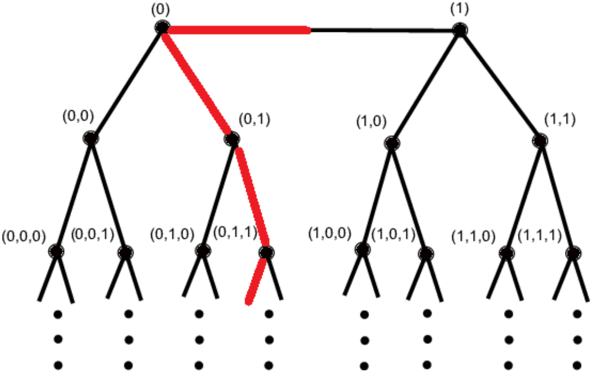
$$\frac{1}{3} = \sum_{j=1}^{\infty} \frac{\varepsilon_j}{3^j} = B, \quad \varepsilon_1 = 0, \quad \varepsilon_j = 2 \quad \text{for} \quad j \geq 2.$$

There is a bijection $\phi : \{0,1\}^{\mathbb{N}} \to \mathcal{C}$ defined by

$$\phi(z) = \frac{2}{3} \sum_{j=0}^{\infty} \frac{z_j}{3^j} \quad \text{for} \quad z = (z_j)_{j \in \mathbb{N}}, \quad z_j \in \{0,1\},$$

and consequently $\mathrm{card}\,(\mathcal{C}) = \mathrm{card}\,(\{0,1\}^{\mathbb{N}}) = \mathrm{card}\,(\mathbb{R}) = \mathfrak{c}.$

Cantor tree



$$\varepsilon = (0, 1, 1, 0, \varepsilon_4, \varepsilon_5, \dots)$$