Lecture 17

Continuous functions,
Continuous functions on compact and connected sets

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Limits

Limits

Let (X, ρ_X), (Y, ρ_Y) be metric spaces. Suppose $E \subseteq X$ and $f : E \to Y$ and p is a limit point of E. We write

$$f(x) \xrightarrow[x \to p]{} q \quad \text{or} \quad \lim_{x \to p} f(x) = q.$$

if there is a point $q \in X$ satisfying the following ε-δ condition:

- For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho_Y(f(x), q) < \varepsilon$$

  for all points $x \in E$ for which $0 < \rho_X(x, p) < \delta$.

Special case

If $X = Y = \mathbb{R}$ then

$$\rho_X(x, y) = \rho_Y(x, y) = |x - y|$$

and the condition reads as follows:

Limit

For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in E$ if

$$0 < |x - p| < \delta,$$

then

$$|f(x) - q| < \varepsilon.$$

Theorem

Theorem (Characterizations of Continuity)

Let (X, ρ_X), (Y, ρ_Y) be metric spaces and $E \subseteq X$, $f : X \to Y$, and $p \in X$ be as in the previous definition. Then

- **A** $\lim_{x \to p} f(x) = q$ iff
- **B** $\lim_{n \to \infty} f(p_n) = q$ for every sequence $(p_n)_{n \in \mathbb{N}}$ in E such that $p_n \neq p$ and $\lim_{n \to \infty} p_n = p$.

Proof (A)$\Longrightarrow$(B). Suppose that (A) holds. Choose $(p_n)_{n \in \mathbb{N}}$ like in condition (B). Let $\varepsilon > 0$ be given, then there exists $\delta > 0$ such that

$$\rho_Y(f(x), q) < \varepsilon \quad \text{if} \quad x \in E \quad \text{and} \quad 0 < \rho_X(x, p) < \delta.$$

Also there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $0 < \rho_X(p_n, p) < \delta$. Thus we also have $\rho_Y(f(p_n), q) < \varepsilon$ for $n \geq N$ showing that (B) holds. □

Proof (B)$\Longrightarrow$(A)

Proof (B)$\Longrightarrow$(A). Conversely suppose (A) is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ) for which

$$\rho_Y(f(x), q) \geq \varepsilon \quad \text{but} \quad 0 < \rho_X(x, p) < \delta.$$

Taking $\delta_n = \frac{1}{n}$ for each $n \in \mathbb{N}$ we thus find a sequence $(p_n)_{n\in\mathbb{N}}$ in E satisfying $\lim_{n\to\infty} p_n = p$ but

$$\rho_Y(f(p_n), q) \geq \varepsilon.$$

thus (B) is false as desired. □

Remark

It was possible to choose the sequence $(p_n)_{n\in\mathbb{N}}$ in E in one step thanks to the Axiom of Choice. Without assuming the Axiom of Choice the previous theorem is not provable.

Theorem

Theorem

Suppose that (X, ρ_X) is a metric space, and $E \subseteq X$, and p is a limit point of E. Let $f, g : E \to \mathbb{R}$ be functions such that

$$\lim_{x \to p} f(x) = A \quad \text{and} \quad \lim_{x \to p} g(x) = B.$$

Then

a. $\lim_{x \to p}(f + g)(x) = A + B$,
b. $\lim_{x \to p}(f \cdot g)(x) = A \cdot B$,
c. $\lim_{x \to p} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$ if $B \neq 0$ and $g(x) \neq 0$ for $x \in E$.

Continuous function

Continuous at the point p

Suppose that (X, ρ_x) and (Y, ρ_Y) are metric spaces, $E \subseteq X$, $p \in E$ and $f : E \to Y$. The function f is said to be **continuous at point** p if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which

$$\rho_X(x, p) < \delta.$$

Continuous function

If the function $f : E \to Y$ is continuous at every point of E then f is said to be **continuous** on E.

Special case

If $X = Y = \mathbb{R}$ then

$$\rho_X(x, y) = \rho_Y(x, y) = |x - y|$$

and the function $f : E \to \mathbb{R}$ is said to be **continuous at point** p if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(p)| < \varepsilon$$

for all points $x \in E$ for which

$$|x - p| < \delta.$$

Example

Example

Let us define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Determine if f is continuous or not at the point 0.

Solution. Let us consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = \sqrt{2}/n$. Then $\lim_{n\to\infty} a_n = 0$ and $a_n \notin \mathbb{Q}$, so $f(a_n) = 0$. Then

$$\lim_{n\to\infty} f(a_n) = 0 \neq 1 = f(0),$$

so f is **not continuous at point** 0. □

Example

Example

Let us define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 \text{ if } x+y \in \mathbb{Q}, \\ 0 \text{ if } x+y \notin \mathbb{Q}. \end{cases}$$

Determine if f is continuous or not at the point $(0,0)$.

Solution. Let us consider the sequence $(a_n)_{n\in\mathbb{N}}$, where $a_n = (0, \sqrt{2}/n)$. Then $\lim_{n\to\infty} a_n = (0,0)$ and $0 + \sqrt{2}/n \notin \mathbb{Q}$, so $f(a_n) = 0$. Then

$$\lim_{n\to\infty} f(a_n) = 0 \neq 1 = f(0,0),$$

so f is **not continuous at point** $(0,0)$. □

Remark

Remark

If p is an isolated point of E then our definition implies that every function f which has E as its domain is continuous at p. For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $e \in E$ for which

$$\rho_X(x, p) < \delta$$

is $x = p$, then

$$\rho_Y(f(x), f(p)) = 0 < \varepsilon.$$

Fact

In the situation of the definition of continuity assume also that p is a limit point of E. Then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

Proof. It is obvious if we compare two previous definitions. □

Theorem

Theorem

Suppose that (X, ρ_X), (Y, ρ_Y), and (Z, ρ_Z) are metric spaces, let $E \subseteq X$ and $f : E \to Y$ and $g : f[E] \to Z$ be given and define $h : E \to Z$ by

$$h(x) = g(f(x)), \quad x \in E.$$

If f is continuous at a point $p \in E$ and g is continuous at the point $f(p)$, then h is continuous at p. In other words

$$\lim_{x \to p} h(x) = \lim_{x \to p} g(f(x)) = g(f(p)) = h(p).$$

Proof

Let $\varepsilon > 0$ be given.

- Since g is continuous at $f(p)$ there is $\eta > 0$ such that

$$\rho_Z(g(y), g(f(p))) < \varepsilon \quad \text{if} \quad \rho_Y(y, f(p)) < \eta \quad \text{and} \quad y \in f[E].$$

- Since f is continuous at p, there is $\delta > 0$ such that

$$\rho_Y(f(x), f(p)) < \eta \quad \text{if} \quad \rho_X(x, p) < \delta \quad \text{and} \quad x \in E.$$

- If follows that

$$\rho_Z(h(x), h(p)) = \rho_Z(g(f(x)), g(f(p))) < \varepsilon$$

 if $\rho_X(x, p) < \delta$ and $x \in E$. Thus h is continuous at $p \in E$. □

Example

Example

Assume that $f : \mathbb{R}^2 \to (0, \infty)$ is continuous for all $(x, y) \in \mathbb{R}^2$. Prove that $h(x, y) = \sqrt{f(x, y)}$ is continuous.

Solution. Let us note that the function $g : (0, \infty) \to (0, \infty)$ defined by

$$g(x) = \sqrt{x}$$

is continuous. We have

$$h = g \circ f,$$

so h is continuous by the previous theorem. □

Theorem

Theorem

A mapping f of a metric space (X, ρ_X) into a metric space (Y, ρ_Y) is continuous on X iff $f^{-1}[V]$ is open in X for every open set V in Y.

Proof. Suppose that f is continuous on X and $V \subseteq Y$ is open.

- We have to show that $f^{-1}[V]$ is open in X. Let $p \in f^{-1}[V]$. Since V is open $B_{\rho_Y}(f(p), \varepsilon) \subseteq V$ for some $\varepsilon > 0$.
- Since f is continuous at $p \in X$ there is $\delta > 0$ such that

$$\rho_Y(f(x), f(p)) < \varepsilon \quad \text{if} \quad \rho_X(x, p) < \delta.$$

  Thus

$$B_{\rho_X}(p, \delta) \subseteq f^{-1}[V] = \{x \in X \ : \ f(x) \in V\}.$$

Proof

Conversely, suppose $f^{-1}[V]$ is open in X for any open $V \subseteq Y$.

- Fix $p \in X$ and $\varepsilon > 0$ and consider
$$V = B_{\rho_Y}(f(p), \varepsilon)$$
which is open thus $f^{-1}[V]$ is open, hence there is $\delta > 0$ so that $B_{\rho_X}(p, \delta) \subseteq f^{-1}[V]$.
- Thus if $\rho_X(x, p) < \delta$, then $x \in f^{-1}[V]$, hence
$$f(x) \in V = B_{\rho_Y}(f(p), \varepsilon) \iff \rho_Y(f(x), f(p)) < \varepsilon. \qquad \square$$

Corollary

A mapping $f : X \to Y$ between metric spaces (X, ρ_X) and (Y, ρ_Y) is continuous iff $f^{-1}[C]$ is closed in X for any closed set C in Y.

Proof. A set is closed iff its complement is open. We are done by invoking the previous theorem, since $f^{-1}[E^c] = (f^{-1}[E])^c$ for every open set $E \subseteq Y$. $\square$

Example

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $a \in R$. Prove that the set

$$A = \{x \in \mathbb{R} \ : \ f(x) > a\}$$

is open.

Solution: We have

$$\{x \in \mathbb{R} \ : \ f(x) > a\} = f^{-1}[(a, \infty)]$$

and (a, ∞) is open in $\mathbb{R}$, so by the previous theorem, A is open. □

Example

Example

Prove that the set

$$A = \{(x, y) \in \mathbb{R} \ : \ \sqrt{x^2 + y^2} < 1\}$$

is open in $\mathbb{R}^2$ with the Euclidean metric.

Solution: Let us consider a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Moreover, by the previous theorem

$$A = \{(x, y) \in \mathbb{R} \ : \ f(x, y) < 1\} = f^{-1}[B(0, 1)]$$

is open since $B(0, 1)$ is an open unit ball in $\mathbb{R}^2$. $\square$

Theorem

Theorem

Let $f, g : X \to \mathbb{R}$ be two continuous functions on a metric space (X, ρ_X). Then $f + g$, $f \cdot g$, and $\frac{f}{g}$ are continuous. In the last case we assume $g(x) \neq 0$ for all $x \in X$.

Example 1

Every polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is a continuous function on $\mathbb{R}$.

Example 2

The exponential function $f(x) = e^x$ is continuous as we have shown that for any $(a_n)_{n\in\mathbb{N}}$ so that $\lim_{n\to\infty} a_n = a$ one has $\lim_{n\to\infty} e^{a_n} = e^a$.

Examples

Example 3

$f(x) = |x|$ is continuous on $\mathbb{R}$ since $|f(x) - f(y)| \le |x - y|$.

Example 4

$f(x) = \lfloor x \rfloor = \max\{n \in \mathbb{Z} \ : \ n \le x\}$ is **NOT** continuous at any $x \in \mathbb{Z}$.

Example 5

$f(x) = x^\alpha$ for any $\alpha \in \mathbb{R}$ is continuous on $(0, \infty)$.

Example 6

If $f, g : X \to \mathbb{R}$ are continuous then $\max\{f, g\}$ and $\min\{f, g\}$ are continuous as well. Indeed,

$$\max\{f, g\} = \frac{f + g + |f - g|}{2}, \qquad \min\{f, g\} = \frac{f + g - |f - g|}{2}.$$

Continuity and compactness

Bounded function

A mapping $f : E \to \mathbb{R}$ is said to be **bounded** if there is a number $M > 0$ such that

$$|f(x)| \leq M \quad \text{for all} \quad x \in E.$$

Theorem (4.4.1)

Suppose that f is a continuous mapping of a compact metric space (X, ρ_X) into a metric space (Y, ρ_Y). Then $f[X]$ is compact in Y.

Proof

Let $(V_\alpha)_{\alpha\in A}$ be an open cover of $f[X]$, i.e.

$$f[X] \subseteq \bigcup_{\alpha\in A} V_\alpha.$$

Since f is continuous then each set $f^{-1}[V_\alpha]$ is open in X. Since X is compact and

$$X \subseteq \bigcup_{\alpha\in A} f^{-1}[V_\alpha]$$

thus there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in A$ so that

$$X \subseteq \bigcup_{j=1}^{n} f^{-1}[V_{\alpha_j}].$$

Since $f[f^{-1}[E]] \subseteq E$ we have

$$f[X] \subseteq f\Big[\bigcup_{j=1}^{n} f^{-1}[V_{\alpha_j}]\Big] \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}. \qquad \square$$

Corollary

Corollary

If $f : X \to \mathbb{R}$ is continuous on a compact metric space (X, ρ_X) then $f[X]$ is closed and bounded in $\mathbb{R}$. Specifically, f is bounded.

Theorem

Suppose $f : X \to \mathbb{R}$ is continuous on a compact metric space (X, ρ_X) and

$$M = \sup_{p \in X} f(p) \quad \text{and} \quad m = \inf_{p \in X} f(p).$$

Then there are p, q such that

$$f(p) = M \quad \text{and} \quad f(q) = m.$$

Proof. $f[X] \subseteq \mathbb{R}$ is closed and bounded. Thus M and m are members of $f[X]$ and we are done. □

Theorem

Theorem

Suppose f is continuous injective mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x, \quad x \in X$$

is a continuous mapping of Y onto X.

Proof. The inverse $f^{-1} : Y \to X$ is well defined since $f : X \to Y$ is one-to-one and onto. It suffices to prove that $f[V]$ is open in Y for every open set V in X. Fix $V \subseteq X$ open, V^c is closed in X thus compact, hence $f[V^c]$ is compact subset of Y and consequently $f[V^c]$ is closed. Since $f : X \to Y$ is one-to-one and onto, hence

$$f[V] = (f[V^c])^c$$

and, consequently, $f[V]$ is open as desired. □

Continuity and connectivity

Theorem

If $f : X \to Y$ is continuous mapping of a metric space X into a metric space Y and if E is a connected subset of X then $f[E]$ is connected in Y.

Proof. Assume for a contradiction that $f[E] = A \cup B$, where A and B are nonempty separated sets in Y. Put

$$G = E \cap f^{-1}[A] \quad \text{and} \quad H = E \cap f^{-1}[B].$$

Then $E = G \cup H$ and neither G nor H is empty.

- Since $A \subseteq \mathrm{cl}\,(A)$ we have $G \subseteq f^{-1}[\mathrm{cl}\,(A)]$ and the latter set is closed since f is continuous hence $\mathrm{cl}\,(G) \subseteq f^{-1}[\mathrm{cl}\,(A)]$.
- Hence

$$f[\mathrm{cl}\,(G)] \subseteq f[f^{-1}[\mathrm{cl}\,(A)]] \subseteq \mathrm{cl}\,(A).$$

Proof

- Since $f[H] \subseteq B$ and $\mathrm{cl}\,(A) \cap B = \emptyset$ we conclude that

$$f[H \cap \mathrm{cl}\,(G)] \subseteq f[\mathrm{cl}\,(G)] \cap f[H] \subseteq \mathrm{cl}\,(A) \cap B = \emptyset,$$

  so $H \cap \mathrm{cl}\,(G) = \emptyset$.
- The same argument shows that $\mathrm{cl}\,(H) \cap G = \emptyset$.
- Thus G and H are separated sets, which is **a contradiction since E is connected**. □

Darboux property

Darboux property (intermediate value theorem)

Let f be a continuous function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there is a point $x \in (a, b)$ such that

$$f(x) = c.$$

A similar result holds if $f(a) > f(b)$.

Proof. $[a, b]$ is connected so $f\big[[a, b]\big]$ is connected in $\mathbb{R}$ as well by the previous theorem. Thus if $f(a) < c < f(b)$, then $c \in f\big[[a, b]\big]$, so there is $x \in [a, b]$ so that $f(x) = c$. □

Remark

The theorem stated above is sometimes called **Darboux property** or **the intermediate value theorem**.

Example

Exercise

Prove that the equation

$$x^3 - x^2 + 2x + 3 = 0$$

has a solution x_0 such that $-1 \leq x_0 \leq 0$.

Solution. Consider a continuous function

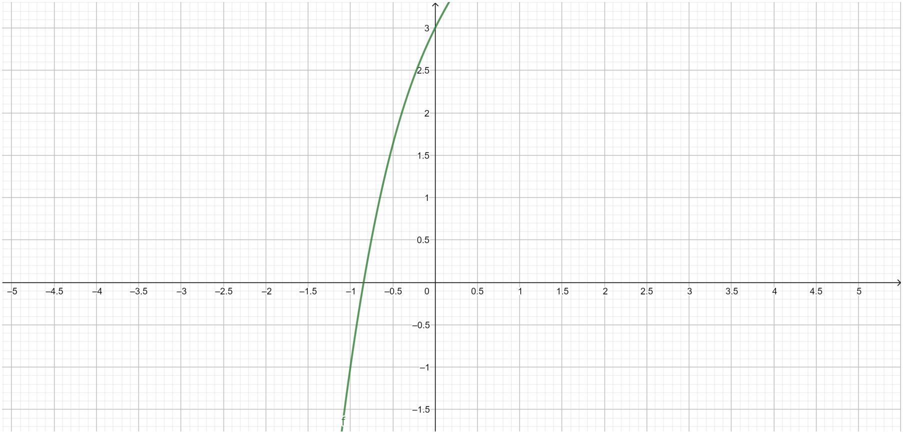
$$f(x) = x^3 - x^2 + 2x + 3.$$

We calculate

$$f(-1) = -1, \quad \text{and} \quad f(0) = 3.$$

It follows by the Darboux property that there is $c \in [-1, 0]$ such that $f(c) = 0$. Thus c is a solution of our equation as desired. □

$f(x) = x^3 - x^2 + 2x + 3,\ x_0 \approx -0.8437$



Example

Exercise

Prove that the equation

$$x^3 = 20 + \sqrt{x}$$

has solution x_0.

Solution. Consider a continuous function

$$f(x) = x^3 - \sqrt{x} - 20.$$

We calculate

$$f(1) = -20 < 0, \quad \text{and} \quad f(4) = 42 > 0.$$

It follows by the Darboux property that there is $c \in [1, 4]$ such that $f(c) = 0$. Thus c is a solution of our equation as desired. □

$f(x) = x^3 - \sqrt{x} - 20,\ x_0 \approx 2,7879$