Lecture 18

Uniform continuity, Banach Contraction Principle, Sets of Discontinuity

MATH 411H, FALL 2025

November 3, 2023

Uniformly continuous mappings

Uniformly continuous mappings

Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces and $f : X \to Y$. We say that f is **uniformly continuous on** X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho_Y(f(x), f(y)) < \varepsilon$$

for all $x, y \in X$ for which

$$\rho_X(x, y) < \delta.$$

Remark

- Uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point.

Remarks

Remark 1

- If f is continuous on X then for each $\varepsilon > 0$ and $p \in X$ there is $\delta > 0$ such that $\rho_X(x, p) < \delta$ implies $\rho_Y(f(x), f(p)) < \varepsilon$.
- Thus $\delta > 0$ **depends on** $p \in X$ **and** $\varepsilon > 0$.

Remark 2

- If f is uniformly continuous on X then for each $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ if $\rho_X(x, y) < \delta$ then $\rho_X(f(x), f(p)) < \varepsilon$.
- Thus $\delta > 0$ **depends only on** $\varepsilon > 0$**, but is uniform for all** $x, y \in X$.

Remark 3

- **Uniform continuity implies continuity.**

Continuity on compact spaces becomes uniform

Theorem

Let f be a continuous mapping of a compact metric space (X, ρ_X) into a metric space (Y, ρ_Y). Then f is uniformly continuous on X.

Proof. Let $\varepsilon > 0$ be given.

- Since f is continuous we can associate to each point $p \in X$ a positive number $\delta_p > 0$ such that if $q \in B(p, \delta_p)$, then $\rho_Y(f(p), f(q)) < \frac{\varepsilon}{2}$.
- Observe that

$$X \subseteq \bigcup_{p \in X} B\left(p, \frac{\delta_p}{2}\right).$$

- Since X is **compact** there are $p_1, p_2, \ldots, p_n \in X$ so that

$$X \subseteq \bigcup_{k=1}^{n} B\left(p_k, \frac{\delta_{p_k}}{2}\right).$$

Proof

- Set

$$\delta = \frac{1}{2} \min \left(\delta_{p_1}, \dots, \delta_{p_n} \right) > 0.$$

- Let $p, q \in X$ be such that $\rho_X(p, q) < \delta$, then there is $1 \leq m \leq n$ such that $p \in B\left(p_m, \frac{\delta_{p_m}}{2}\right)$. Hence

$$\rho_X(q, p_m) \leq \rho_X(q, p) + \rho_X(p_m, p) \leq \delta + \frac{\delta_{p_m}}{2} < \delta_{p_m}.$$

- Thus we conclude

$$\rho_Y(f(p), f(q)) \leq \rho_Y(f(p), f(p_m)) + \rho_Y(f(p_m), f(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof. □

Example

Exercise

Let $f(x) = \frac{1}{\sqrt{x}}$. Determine if it is uniformly continuous on $[1, 2]$.

Solution. The interval $[1, 2]$ is compact and the function f is continuous at every point of $[1, 2]$. Hence, by the previous theorem, it is **uniformly continuous**. □

Exercise

Let

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

Determine if it is uniformly continuous on $[1, 2]$.

Solution. The interval $[1, 2]$ is compact and the function f is continuous at every point of $[1, 2]$. Hence, by the previous theorem, it is **uniformly continuous**. □

Example

Exercise

Let

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

Determine if it is uniformly continuous on $[0, 1]$.

Solution. Let us consider $a_n = \frac{1}{n}$. Then $\lim_{n\to\infty} a_n = 0$, but

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} n \neq f(0) = 0,$$

so f is not continuous at the point 0, so it is **not uniformly continuous**. □

Example

Exercise

Show that the function

$$f(x) = \begin{cases} \frac{1}{x} \text{ if } x \neq 0, \\ 0 \text{ if } x = 0, \end{cases} \qquad \text{is not uniformly continuous on } (0,1).$$

Solution. It can be checked that f is continuous on $(0,1)$.

- Suppose that f is uniformly continuous, then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in (0,1)$ if $|x-y| < \delta$ then
$$|f(x) - f(y)| < \varepsilon.$$
- We will use this condition with $\varepsilon = 1$ and $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$.
- This leads to a contradiction, since if $\frac{1}{n} < \delta$, then we see that

$$|x-y| = \frac{1}{n(n+1)} < \delta \quad \text{implies} \quad 1 = |n-n+1| = |f(x)-f(y)| < 1.$$

Lipschitz mapping

Lipschitz mapping

Let (X, ρ) be a metric space. We say that $\phi : X \to X$ is **a Lipschitz mapping of X into itself** with the **Lipschitz constant** $C_\phi > 0$ if it satisfies

$$\rho(\phi(x), \phi(y)) \leq C_\phi \rho(x, y) \quad \text{for all} \quad x, y \in X.$$

Remark

Every Lipschitz mapping is uniformly continuous.

Example 1

Let $X = \mathbb{R}$ and $\phi(x) = ax + b$, then ϕ is a Lipschitz map with $C_\phi = |a|$ since

$$|\phi(x) - \phi(y)| = |a||x - y| \quad \text{for all} \quad x, y \in \mathbb{R}.$$

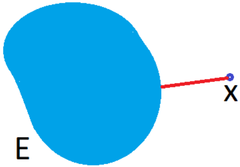
Lipschitz mappings - examples

Example 2

Let (X, ρ) be a metric space and $\emptyset \neq E \subseteq X$.

- Define **the distance from** $x \in X$ **to** E by setting

$$\rho_E(x) = \inf\{\rho(x, z) \ : \ z \in E\}.$$



- Sometimes we write $\rho_E(x) = \rho(x, E)$.

Example

- One can easily verify that $\rho(x, E) = 0$ iff $x \in \mathrm{cl}\,(E)$.
- Moreover,
$$|\rho(x, E) - \rho(y, E)| \leq \rho(x, y),$$
thus $X \ni x \to \rho(x, E)$ is Lipschitz with the Lipschitz constant 1.
- Indeed, for any $z \in E$ we have
$$\rho(x, E) \leq \rho(x, z) \leq \rho(x, y) + \rho(y, E),$$
so
$$\rho(x, E) - \rho(y, E) \leq \rho(x, y).$$
By symmetry $\rho(y, E) - \rho(x, E) \leq \rho(x, y)$, and we are done. □

Contraction

Contraction

Let (X, ρ) be a metric space. Suppose that $\phi : X \to X$ and there is $c \in (0, 1)$ such that

$$\rho(\phi(x), \phi(y)) \leq c\rho(x, y) \quad \text{for all} \quad x, y \in X$$

then ϕ is said to be **a contraction** of X into X.

Remark

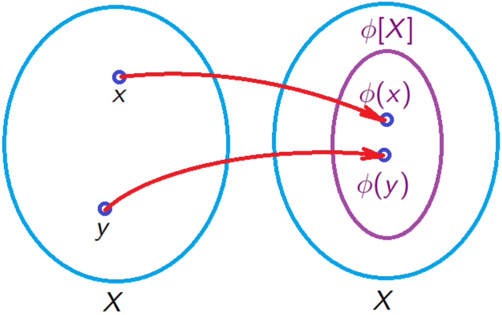
In other words, contractions $\phi : X \to X$ are Lipschitz maps with the Lipschitz constants

$$L_\phi < 1.$$

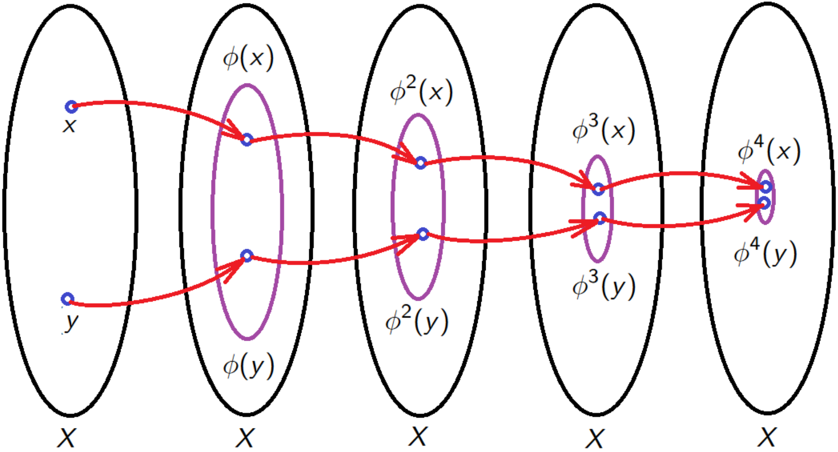
The Banach contraction principle

The Banach contraction principle

If (X, ρ) is a complete metric space and if ϕ is a contraction of X into X, then there exists one and only one $x \in X$ such that $\phi(x) = x$



Idea of the proof



Proof: 1/2

Proof of the uniqueness. If there are $x, y \in X$ so that $x \neq y$ and $\phi(x) = x$ and $\phi(y) = y$, then

$$0 < \rho(x, y) = \rho(\phi(x), \phi(y)) \leq c\rho(x, y) < \rho(x, y),$$

which is a contradiction. Thus we must have $x = y$. □

Remark

The Banach contraction principle says that ϕ has a unique fixed point.

Proof of the existence. The existence of a fixed point of $\phi : X \to X$ is the essential part of the proof. The proof furnishes **a construction method** for looking for the fixed point.

- Pick x_0 arbitrarily and consider

$$x_1 = \phi(x_0), \qquad x_2 = \phi(x_1) = \phi^2(x_0), \qquad x_3 = \phi(x_2) = \phi^3(x_0), \ldots$$

$$x_{n+1} = \phi(x_n) = \phi^{n+1}(x_0) \quad \text{for} \quad n \in \mathbb{N}.$$

Proof: 2/2

- Observe that $\rho(x_{n+1}, x_n) = \rho(\phi(x_n), \phi(x_{n-1})) \leq c\phi(x_n, x_{n-1})$.
- Thus inductively we obtain $\rho(x_{n+1}, x_n) \leq c^n \rho(x_1, x_0)$ for $n \in \mathbb{N}$.
- If $n < m$ it follows that

$$\rho(x_n, x_m) \leq \sum_{j=n+1}^{m} \rho(x_j, x_{j+1}) \leq \left(c^n + c^{n+1} + \ldots + c^{m-1}\right) \rho(x_1, x_0)$$

$$\leq c^n(1 + c + c^2 + \ldots)\rho(x_0, x_1) = \frac{c_n}{1-c}\rho(x_0, x_1) \xrightarrow[n\to\infty]{} 0.$$

- Thus the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
- But X is **complete metric space** so there is $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \quad \text{for some} \quad x \in X.$$

Since $\phi : X \to X$ is continuous, thus

$$\phi(x) = \lim_{n\to\infty} \phi(x_n) = \lim_{n\to\infty} x_{n+1} = x. \qquad \square$$

Example

Exercise

Consider $f : [1, \infty) \to [1, \infty)$ defined by

$$f(x) = \frac{x}{4} + \frac{1}{4x}.$$

Prove that f has an unique fixed point.

Solution. Note that f is a contraction. We have

$$|f(x) - f(y)| = \left| \frac{x}{4} + \frac{1}{4x} - \frac{y}{4} - \frac{1}{4y} \right| \leq \frac{1}{4}|x - y| + \frac{1}{4}\left| \frac{1}{x} - \frac{1}{y} \right| \leq \frac{|x - y|}{2}.$$

By the Banach contraction principle, f has a unique fixed point. □

Example

Exercise

Prove that there is a unique $x \geq 0$ such that

$$x = \sqrt{2+x}.$$

Solution. Consider $f : [0, \infty) \to [0, \infty)$ defined by

$$f(x) = \sqrt{x+2}.$$

By the Banach contraction principle, it suffices to prove that f is a contraction. Indeed,

$$\left|\sqrt{x+2} - \sqrt{y+2}\right| = \frac{|x-y|}{\sqrt{x+2} + \sqrt{2+y}} \leq \frac{1}{2\sqrt{2}}|x-y|. \quad \square$$

Example

Exercise

Prove that the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}},\ldots$$

converges.

Solution. As in the proof of the Banach contraction principle, the sequence defined by

$$x_0 = \sqrt{2} \quad \text{and} \quad x_{n+1} = \sqrt{x_n + 2} \quad \text{for } n \in \mathbb{N}$$

converges to the unique solution of

$$x = \sqrt{2+x}.$$

Since $f(x) = \sqrt{x+2}$ is a contraction we are done. □

Discontinuities

Discontinuities

If x is a point in the domain of a function f at which f is not continuous we say that

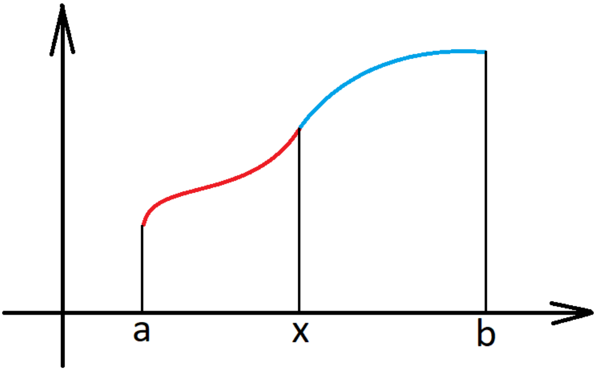
- f **is discontinuous** on X,
- or f **has a discontinuity at** $x \in X$.

Definition

Let $f : (a, b) \to \mathbb{R}$. Consider any x such that $a < x < b$.

- We write $f(x+) = q$ if $f(t_n) \xrightarrow[n\to\infty]{} q$ for all sequences $(t_n)_{n\in\mathbb{N}}$ in (x, b) such that $t_n \xrightarrow[n\to\infty]{} x$.
- Similarly, $f(x-) = q$ if $f(t_n) \xrightarrow[n\to\infty]{} q$ for all sequences $(t_n)_{n\in\mathbb{N}}$ in (a, x) such that $t_n \xrightarrow[n\to\infty]{} x$.
- It is clear that for any $x \in (a, b)$ $\lim_{t\to x} f(t)$ exists iff $f(x+) = f(x-) = \lim_{t\to x} f(t)$.

$f(x+)$ and $f(x-)$ - picture



Discontinuity of first and second kind

Let $f : (a, b) \to \mathbb{R}$ be given.

Discontinuity of the first kind

If f is discontinuous at a point x and if $f(x+)$ and $f(x-)$ exist, then f is said to have **discontinuity of the first kind** or **simple discontinuity at** x.

Discontinuity of the second kind

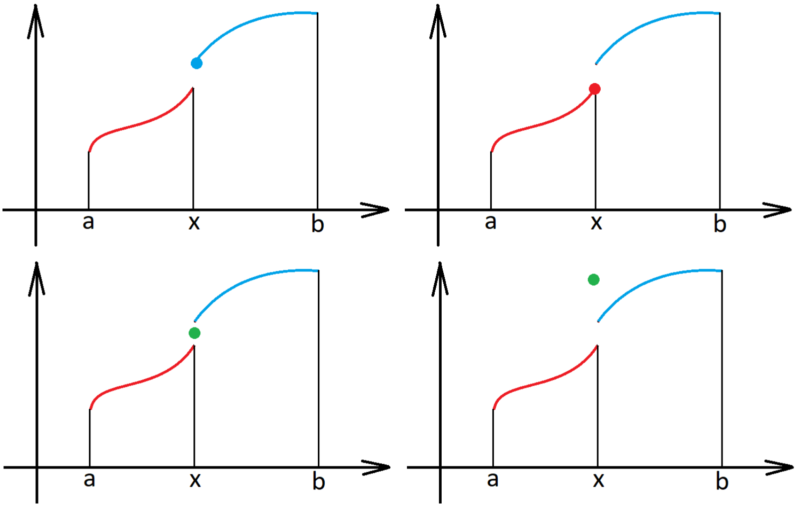
Otherwise the discontinuity is said to be **of the second type**.

Remark

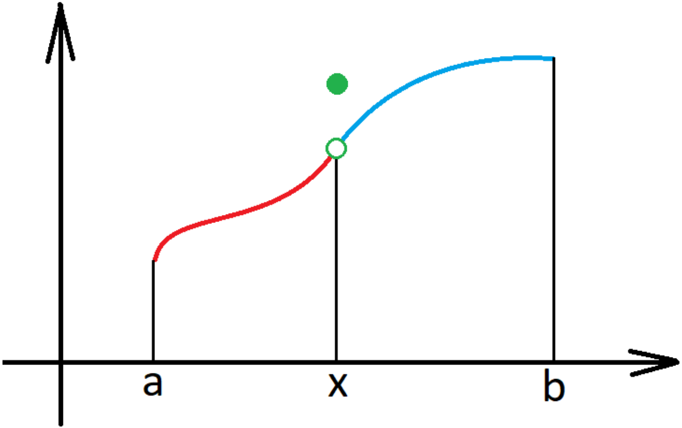
There are two ways in which a function can have a simple discontinuity:

- either $f(x+) \neq f(x-)$,
- or $f(x+) = f(x-) \neq f(x)$.

$f(x+) \neq f(x-)$



$f(x+) = f(x-) \neq f(x)$



Continuous from the left and from the right

Continuous from the left

If $f(x-) = f(x)$ for all $x \in (a, b)$ then we say that f is **continuous from the left**.

Continuous from the right

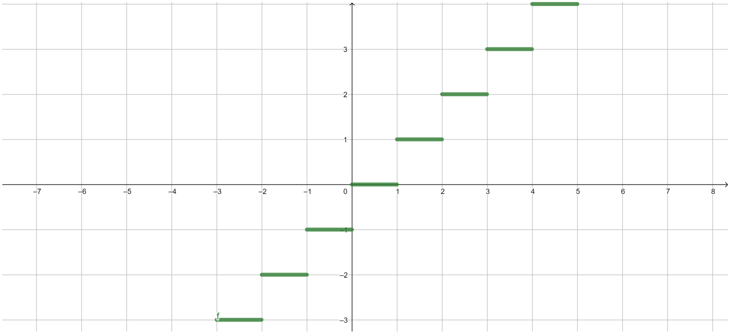
If $f(x+) = f(x)$ for all $x \in (a, b)$ then we say that f is **continuous from the right**.

Integer part

Integer part

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$$

is continuous from the right.

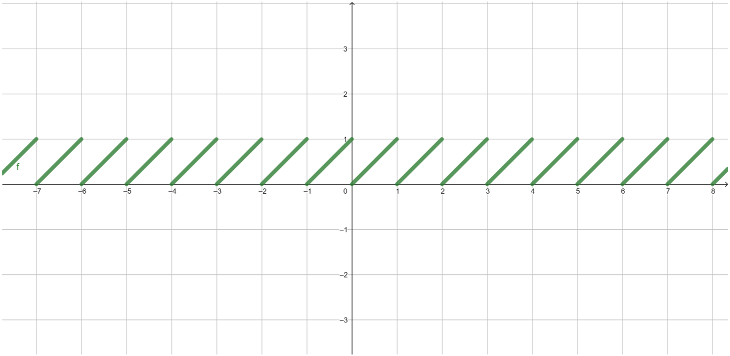


Fractional part

Fractional part

$$\{x\} = x - \lfloor x \rfloor$$

is also continuous from the right.



Examples involving characteristic function of $\mathbb{Q}$

Characteristic function of $\mathbb{Q}$

The function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

has a discontinuity of the second kind at every point x since neither $f(x+)$ nor $f(x-)$ exists.

Characteristic function of $\mathbb{Q}$ times linear function

Define

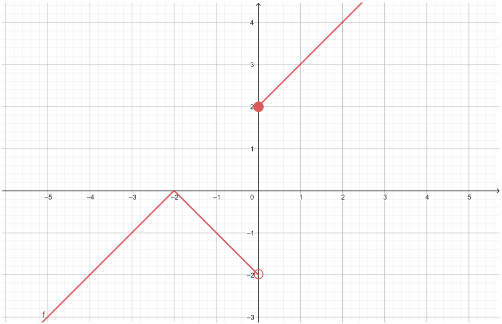
$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then f is continuous at $x = 0$, and f has a discontinuity of the second kind at every other point x since neither $f(x+)$ nor $f(x-)$ exists.

Example

An example of a function with a simple discontinuity at $x = 0$ that is continuous at every other point is given by the following formula

$$f(x) = \begin{cases} x+2 & \text{if } x < -2, \\ -x-2 & \text{if } x \in [-2,0), \\ x+2 & \text{if } x \geq 0. \end{cases}$$



Monotonically increasing and decreasing functions

Monotonically increasing (and decreasing) function

Let $f : (a, b) \to \mathbb{R}$, then f is said to be **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If $f(x) \geq f(y)$ we obtain the definition of **a monotonically decreasing function**.

Theorem

Let f be a monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point at $x \in (a, b)$. More precisely,

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) \leq \inf_{x<t<b} f(t).$$

Furthermore, if $a < x < y < b$ then $f(x+) \leq f(y-)$. Analogous result remains true for monotonically decreasing functions.

Proof: 1/2

- The set
$$E = \{f(t) \;:\; a < t < x\}$$
is bounded by $f(x)$ hence $A = \sup E \in \mathbb{R}$ and $A \leq f(x)$.
- We have to show $f(-x) = A$.
- Let $\varepsilon > 0$ be given. Since $A = \sup E$ there is $\delta > 0$ such that $a < x - \delta < x$ and $A - \varepsilon < f(x - \delta) \leq A$. Since f is monotonic
$$f(x-\delta) \leq f(t) \leq A \quad \text{for} \quad t \in (x - \delta, x).$$
- Thus $A - \varepsilon < f(t) \leq A$, so
$$|f(x) - A| < \varepsilon \quad \text{for} \quad t \in (x - \delta, x).$$
- Thus $A = f(x-)$. In a similar way we prove $f(x+) = \inf_{x<t<b} f(t)$.

Proof: 2/2

- Next if $a < x < y < b$, then

$$f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t).$$

- Similarly

$$f(y-) = \sup_{a<t<y} f(t) = \sup_{x<t<y} f(t).$$

- Thus

$$f(x+) = \inf_{x<t<y} f(t) \leq \sup_{x<t<y} f(t) = f(y-). \qquad \square$$

Corollary

Monotonic functions have no discontinuities of the second kind.

Theorem

Theorem

Let $f : (a, b) \to \mathbb{R}$ be monotonic. Then the set of points of (a, b) of which f is discontinuous is at most countable.

Proof. Wlog we may assume that f is increasing.

- Let E be the set of points at which f is discontinuous.
- With every point $x \in E$ we associate a rational number $r(x) \in \mathbb{Q}$ such that

$$f(x-) < r(x) < f(x+),$$

  so $r : E \to \mathbb{Q}$.
- Since $x_1 < x_2$ implies $f(x_1+) \le f(x_2-)$ we see that $r(x_1) \ne r(x_2)$ if $x_1 \ne x_2$.
- We have established that the function $r : E \to \mathbb{Q}$ is injective, thus

$$\mathrm{card}\,(E) \le \mathrm{card}\,(\mathbb{Q}) = \mathrm{card}\,(\mathbb{N}). \qquad \square$$

Proof - illustration