Lecture 1

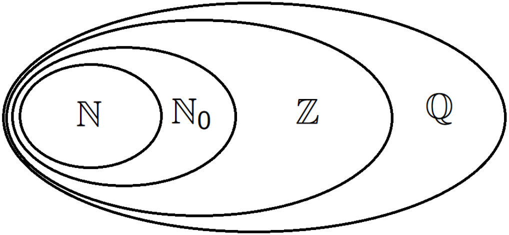
Introduction and basic set theory

MATH 411H, FALL 2025

September 4, 2025

Number systems

- $\mathbb{N} = \mathbb{Z}_+ = \{1, 2, 3, \dots\}$ - positive integers,
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ - non-negative integers,
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$ - the set of integers,
- $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}\}$ - the set of rationals.



- Let $a, d \in \mathbb{Z}$ and we say that d is a divisor of a, and write $d \mid a$, if there exists an integer $q \in \mathbb{Z}$ such that $a = dq$.
- An integer $n \in \mathbb{Z}$ is called prime if $n > 1$ and if the only positive divisors of n are 1 and n. The set of all prime numbers will be denoted by $\mathbb{P}$.

Sets

The words **family** and **collection** will be used synonymously with "set".

Notation

$\emptyset$ - **empty set**,
$\mathcal{P}(X)$ - family of subsets of the set X, sometimes called **power set** of X.

Example 1

If $X = \{1\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{1\}\}.$$

Example 2

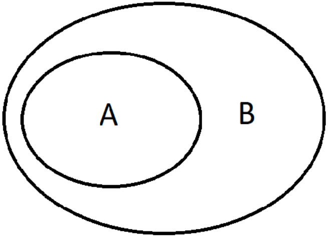
If $X = \{1, 2, 3\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Inclusions 1/2

Definition (Inclusion in a weak sense)

We write $A \subseteq B$ if any element of A is also the element of B.



- We will write $A \subset B$ if $A \subseteq B$ and $A \neq B$.
- In practice, if one wants to prove that $A = B$, it suffices to show that $A \subseteq B$ and $B \subseteq A$ hold simultaneously.

Inclusions 2/2

Example 1

We have $\mathbb{P} \subset \mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q}$.

Example 2

If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then $A \subseteq B$ and $A \subset B$.

Example 3

If $A = \{1, 2, 4\}$ and $B = \{1, 2, 3\}$, then $A \subseteq B$ does not hold, because 4 belongs to A, but it does not belong to B.

Union of sets 1/2

Union of sets

Let X be a set, Σ be a family of sets from $\mathcal{P}(X)$. **The union of the members from** Σ is the following subset of X:

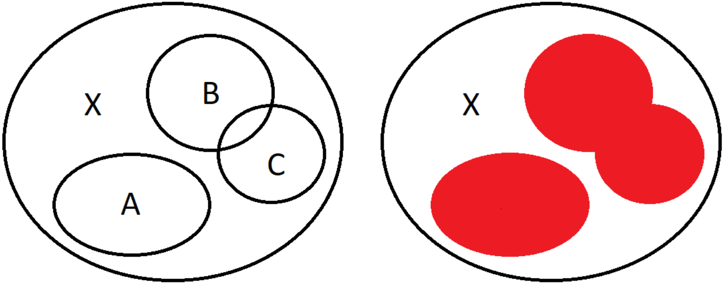
$$\bigcup_{E\in\Sigma} E = \{x \in X \ : \ x \in E \text{ for some } E \in \Sigma\} = \{x \in X \ : \ \exists_{E\in\Sigma} x \in E\}.$$

$\exists \equiv$ there exists.

Union of sets 2/2

Example

If $\Sigma = \{A, B, C\}$, then $\bigcup_{E\in\Sigma} E = A \cup B \cup C$



Intersection of sets 1/3

Intersection of sets

Let X be a set, $\Sigma \neq \emptyset$ be a family of sets from $\mathcal{P}(X)$. **The intersection of the members from Σ** is the following subset of X:

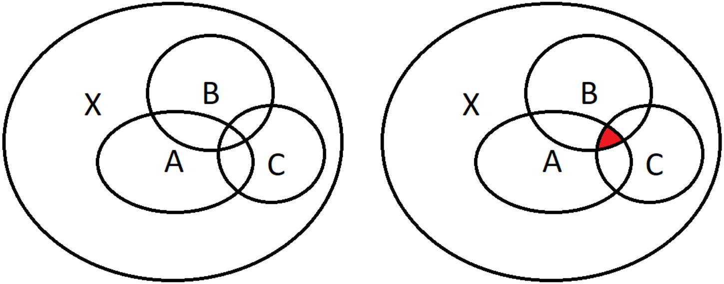
$$\bigcap_{E \in \Sigma} E = \{x \in X \ : \ x \in E \text{ for all } E \in \Sigma\} = \{x \in X \ : \ \forall_{E \in \Sigma} x \in E\}.$$

$\forall \equiv$ for all.

Intersection of sets $2/3$

Example 1

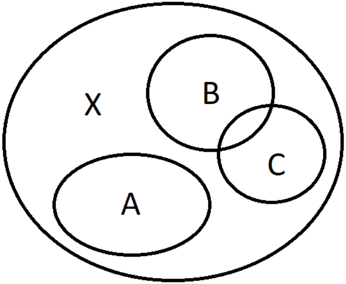
If $\Sigma = \{A, B, C\}$, then $\bigcap_{E \in \Sigma} E = A \cap B \cap C$



Intersection of sets 3/3

Example 2

If $\Sigma = \{A, B, C\}$ as in the picture, then $\bigcap_{E\in\Sigma} E = A \cap B \cap C = \emptyset$.



Union and intersection of indexed family of sets

If $\Sigma = \{E_\alpha : \alpha \in A\}$, then the union and the intersection will be denoted respectively by

$$\bigcup_{\alpha\in A} E_\alpha \text{ and } \bigcap_{\alpha\in A} E_\alpha.$$

Example 1

If $A = \{1, 2, 3\}$, then $\bigcup_{\alpha\in A} E_\alpha = E_1 \cup E_2 \cup E_3$.

Example 2

If $A = \mathbb{N}$, then $\bigcup_{\alpha\in A} E_\alpha = E_1 \cup E_2 \cup E_3 \cup E_4 \cup \ldots$.

Disjointness

Definition (Disjointness)

If $A \cap B = \emptyset$, then we say that A and B are **disjoint**.

Example

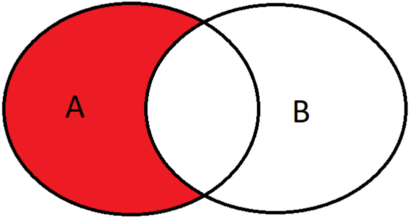
If $A = \{1, 2\}$, $B = \{3, 4\}$, $C = \{1, 2, 3\}$, then A and B are disjoint, but A and C are not disjoint.

Difference of sets

Difference of sets

If A, B are two sets, then

$$A \setminus B = \{x \in A \ : \ x \notin B\}.$$



Example 1

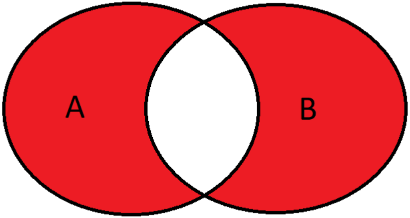
If $A = \{1, 2, 3\}$ and $B = \{3\}$, then $A \setminus B = \{1, 2\}$.

Symmetric difference of sets

Symmetric difference of sets

If A, B are two sets, then

$$A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$



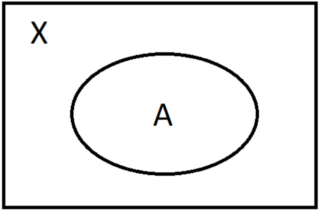
Example

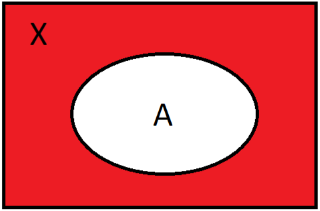
If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then $A \triangle B = \{1, 2, 5, 6\}$.

Complement of sets

Complement of sets

If a set X is given, and $A \subseteq X$, then the complement of A in X is defined by $A^c = X \setminus A$.





de Morgan's laws

de Morgan's laws

$$\left(\bigcup_{\alpha\in A} E_\alpha\right)^c = \bigcap_{\alpha\in A} E_\alpha^c$$

$$\left(\bigcap_{\alpha\in A} E_\alpha\right)^c = \bigcup_{\alpha\in A} E_\alpha^c$$

Example

We have $(A\cup B)^c = A^c \cap B^c$ and $(A\cap B)^c = A^c \cup B^c$.

Ordered pairs

Ordered pairs

The ordered pair (x, y) is precisely the set $\{\{x\}, \{x, y\}\}$.

Theorem

$(x, y) = (u, v)$ iff $x = u$ and $y = v$.

Proof

- If $x = u$ and $y = v$, then $(x, y) = \{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\} = (u, v)$.
- Suppose that $(x, y) = (u, v)$. This is equivalent to say that

$$(x, y) = \{\{x\}, \{x, y\}\} = \{\{u\}, \{u, v\}\} = (u, v).$$

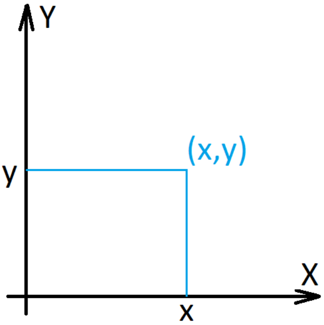
- This implies that $\{x\} = \{u\}$ and $\{x, y\} = \{u, v\}$.
- Hence $x = u$ and $y = v$ as desired. □

Cartesian products

Cartesian products

If X and Y are sets, their **Cartesian product** $X \times Y$ is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

$$X \times Y = \{(x, y) \; : \; x \in X, y \in Y\}$$



Cartesian products - examples

Example 1

If $X = \{1, 2, 3\}$, $Y = \{4, 5\}$, then

$$X \times Y = \{(1,4),(2,4),(3,4),(1,5),(2,5),(3,5)\}.$$

Example 2

If $X = \{1, 2\}$, $Y = \{1, 2\}$, then

$$X \times Y = \{(1,1),(1,2),(2,1),(2,2)\}.$$

Example 3

If $X \neq \emptyset$ and $Y = \emptyset$, then $X \times Y = \emptyset$.

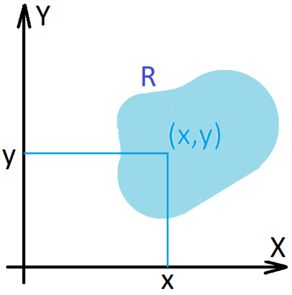
Relations

Relations

A **relation** from X to Y is a subset R of $X \times Y$, i.e. $R \subseteq X \times Y$.

If $X = Y$ we speak about relations on X.

If R is a relation from X to Y we shall sometimes write xRy to mean that $(x, y) \in R \subseteq X \times Y$.



Relations - examples

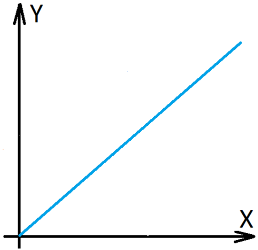
Example 1

If $X = Y$ and we set

$$xRy \iff x = y$$

This relation corresponds to the diagonal Δ in $X \times X$:

$$\Delta = \{(x, x)\ :\ x \in X\} \subseteq X \times X.$$



Now we present more examples of relations.

More examples: functions and sequences

Functions

A **function** $f : X \to Y$ is a relation R from X to Y with the property that for every $x \in X$ there is a unique element $y \in Y$ such that xRy in which case we write

$$y = f(x).$$

Sequences

A **sequence** in X is a function from the natural numbers $\mathbb{N}$ into the set X. That is, it is an assignment of elements from X to natural numbers.

- We usually denote such a function by $\mathbb{N} \ni n \mapsto x_n \in X$, so the terms in the sequence are written $(x_1, x_2, x_3, \ldots)$.
- To refer to the whole sequence, we will write $(x_n)_{n=1}^{\infty}$, or $(x_n)_{n\in\mathbb{N}}$ or for the sake of brevity simply (x_n).

Equivalence relations

Equivalence relations

An **equivalence relation** is a relation on X such that:

1. xRx for all $x \in X$, (reflexivity).
2. xRy iff yRx for all $x, y \in X$, (symmetry).
3. if xRy and yRz, then xRz for all $x, y, z \in R$. (transitivity).

Equivalence classes

An **equivalence class** of an element $x \in X$ is the set $[x] = \{y \in X : xRy\}$.

Observe that $[x] \neq \emptyset$ for every $x \in X$, since R is reflexive.

Properties of equivalence relations

Theorem

Let X be a set, with an equivalence relation R on X. Then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$ for any $x, y \in X$.

Proof

Let $x, y \in X$ and assume that there is some element $z \in [x] \cap [y]$; in other words, xRz and yRz. Now, let $u \in [x]$. Since xRu and xRz then uRz by symmetry and transitivity. But yRz, so again by symmetry and transitivity yRu, which means that $u \in [y]$. We have proved that $[x] \subseteq [y]$. Similarly we obtain the other inclusion $[y] \subseteq [x]$. Hence, $[x] = [y]$ if $[x] \cap [y] \neq \emptyset$.

As an easy consequence we obtain the following important result.

Theorem

X is the disjoint union of the equivalence classes.

Equivalence relations - examples 1/2

Example

Let $X = \mathbb{Z}$. Consider

$$xRy \iff x \equiv y \mod 5 \iff 5|(x-y).$$

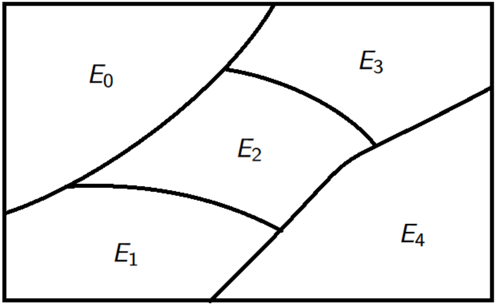
the equivalence classes corresponding to the relation R are the sets:

$$E_0 = [0] = \{y \in \mathbb{Z} : 5|(0-y)\} = \{5k \ : \ k \in \mathbb{Z}\},$$
$$E_1 = [1] = \{y \in \mathbb{Z} : 5|(1-y)\} = \{5k+1 \ : \ k \in \mathbb{Z}\},$$
$$E_2 = [2] = \{y \in \mathbb{Z} : 5|(2-y)\} = \{5k+2 \ : \ k \in \mathbb{Z}\},$$
$$E_3 = [3] = \{y \in \mathbb{Z} : 5|(3-y)\} = \{5k+3 \ : \ k \in \mathbb{Z}\},$$
$$E_4 = [4] = \{y \in \mathbb{Z} : 5|(4-y)\} = \{5k+4 \ : \ k \in \mathbb{Z}\}.$$

Equivalence relations - examples 2/2

We have

$$\mathbb{Z} = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4.$$



Partially ordered sets

Partial ordering

A **partial ordering** on a nonempty set X is a relation R on X with the following properties:

(a) xRx for all $x \in X$, (reflexivity).

(b) If xRy and yRx, then $x = y$, (antisymmetry).

(c) If xRy and yRz, then xRz, (transitivity).

Linear ordering

If R additionally satisfies that for all $x, y \in X$ either xRy or yRx, then R is called **linear** or **total ordering** on X.

Example

The set of rational numbers $\mathbb{Q}$ with the natural order $\leq$ is totally ordered set. We say that $r \leq s$ for $r, s \in \mathbb{Q}$ iff $s - r \geq 0$.

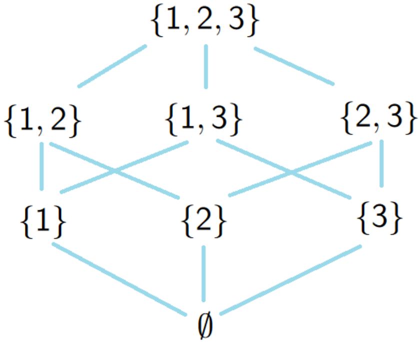
Examples of partial ordering

Example

If X is any set then $P(X)$ is partially ordered by inclusion, i.e.

$$ARB \iff A \subseteq B.$$

Consider $X = \{1, 2, 3\}$ and we have its Hasse diagram



Poset $\equiv$ partially ordered set

Poset

We say that $(X, \leq)$ is a **poset** if the relation "$\leq$" is a partial ordering on X or $(X, \leq)$ is partially ordered by "$\leq$".

- We will write $x < y$ in a poset $(X, \leq)$ iff $x \leq y$ and $x \neq y$.

Upper (lower) bound

Let $(X, \leq)$ be a poset and $A \subseteq X$. An element $x \in X$ is an **upper bound** of A (resp. **lower bound** of A) if $a \leq x$ for all $a \in A$ (resp. $x \leq a$ for all $a \in A$). **An upper (lower) bound $x \in X$ need not belong to A.**

Maximal (minimal) element

Let $(X, \leq)$ be a poset. A **maximal** (resp. **minimal**) element of X is an element $x \in X$ such that if $y \in X$ and $x \leq y$ (resp. $x \geq y$) then $x = y$.

Greatest (least) element

Let $(X, \leq)$ be a poset. A **greatest** (resp. **least**) element of X is an element $x \in X$ such that $y \leq x$ for all $y \in X$ (resp. $x \leq y$ for all $y \in X$).

Remark

Remark

In linearly (totally) ordered sets in contrast to general partially ordered sets

- the greatest and maximal elements **are the same**,
- the least and minimal elements **are the same**.

There may be many maximal and minimal elements in general partially ordered sets, and the maximal (minimal) elements are **not comparable**.

Example

- In order to see this we consider $X = \mathcal{P}(\{1,2,3,4\}) \setminus \{\{1,2,3,4\}\}$. The element $\{1,2,3,4\}$ is an upper bound for X.
- The set X does not have the greatest element, but the elements $\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$ are maximal.
- The empty set $\emptyset$ is both the least and the minimal element for X. The empty set $\emptyset$ is a lower bound for X.