Lecture 20

Exponential Function and Natural Logarithm Function, Power Series and Taylor's theorem

MATH 411H, FALL 2025

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The exponential function

The exponential function

We define

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \text{for} \quad z \in \mathbb{C}.$$

- Observe that $|E(z)| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} < \infty$. The ratio test shows that the series converges absolutely for any $z \in \mathbb{C}$ and $E(z)$ is well defined.

Recall

If $\sum_{n=0}^{\infty} a_n$ converges absolutely, $\sum_{n=0}^{\infty} a_n = A$, and $\sum_{n=0}^{\infty} b_n = B$, and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Then $\sum_{k=0}^{\infty} c_k = AB$.

Properties of the exponential function 1/4

Applying this result to absolutely convergent series $E(z)$, $E(w)$ we obtain

(*)

$$E(z)E(w) = E(z+w) \quad \text{for} \quad z, w \in \mathbb{C}.$$

Proof of (*). Indeed,

$$E(z)E(w) = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) \underbrace{=}_{\text{Recall}} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = E(z+w).$$

In the last line we have used the Binomial theorem. □

Properties of the exponential function 2/4

As the consequence we obtain

(**)

$$E(z)E(-z) = E(z - z) = E(0) = 1.$$

- This shows that $E(z) \neq 0$ for all $z \in \mathbb{C}$.
- We have $E(x) > 0$ if $x > 0$, giving $E(x) > 0$ for all $x \in \mathbb{R}$ by (**).
- It is easy to see that

$$\lim_{x \to \infty} E(x) = +\infty \quad \text{since} \quad E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- Consequently by (**) we obtain

$$\lim_{x \to \infty} E(-x) = 0 \quad \text{since} \quad E(-x) = \frac{1}{E(x)}.$$

Properties of the exponential function 3/4

- If $0 < x < y$ then
$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} < \sum_{n=0}^{\infty} \frac{y^n}{n!} = E(y).$$
- Since $E(x)E(-x) = 1$ thus
$$E(-y) < E(-x),$$
hence E is strictly increasing on $\mathbb{R}$.
- If $x \in \mathbb{R}$ then
$$E'(x) = \lim_{h\to 0} \frac{E(x+h) - E(x)}{h} = E(x) \underbrace{\lim_{h\to 0} \frac{E(h) - 1}{h}}_{=1} = E(x).$$

Properties of the exponential function 4/4

- Indeed,

$$\frac{E(h)-1}{h} = \frac{1}{h}\sum_{n=1}^{\infty}\frac{h^n}{n!} = \sum_{n=1}^{\infty}\frac{h^{n-1}}{n!},$$

hence

$$\left|\frac{1}{h}(E(h)-1)-1\right| \leq \sum_{n=2}^{\infty}\frac{|h|^{n-1}}{n!} = |h|\sum_{n=2}^{\infty}\frac{|h|^{n-2}}{n!}$$
$$\leq |h|E(|h|) \underbrace{\leq}_{|h|\leq 1} |h|e \xrightarrow[h\to 0]{} 0.$$

- We have proved that $E'(x) = E(x)$ for all $x \in \mathbb{R}$.
- In particular, E is continuous on $\mathbb{R}$.

In the next theorem we summarize what we have proved.

Theorem

Theorem

The function

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is called the **exponential function** and is usually denoted by $e^x = E(x)$. The exponential function $\mathbb{R} \ni x \mapsto e^x$ satisfies the following properties:

(a) e^x is continuous and differentiable for all $x \in \mathbb{R}$,

(b) $(e^x)' = e^x$,

(c) e^x is strictly increasing on $\mathbb{R}$ and $e^x > 0$ for all $x \in \mathbb{R}$,

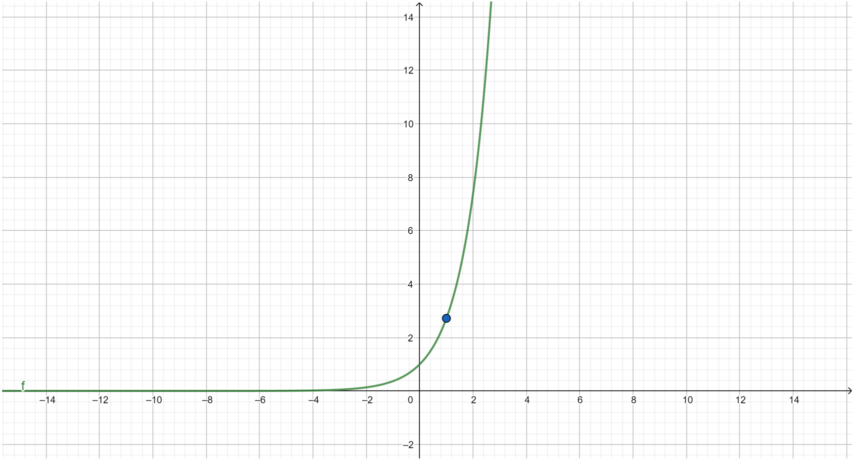
(d) $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$,

(e) $\lim_{x\to+\infty} e^x = +\infty$ and $\lim_{x\to-\infty} e^x = 0$,

(f) $\lim_{x\to+\infty} x^n e^{-x} = 0$ for all $n \in \mathbb{N}$.

Proof. We have proved (a)-(e). We only prove (f).

Graph of $f(x) = e^x$



Proof of (f)

Note that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} > \frac{x^{n+1}}{(n+1)!},$$

so that

$$x^n e^{-x} < \frac{(n+1)!}{x} \xrightarrow[x\to\infty]{} 0,$$

which gives the desired claim. □

Remark

Item (f) says that e^x tends to $+\infty$ faster that any polynomial.

- If $P(x) = \sum_{k=0}^{n} c_k x^k$, where $c_1, \ldots, c_n \in \mathbb{R}$, then

$$0 \leq \left| \frac{P(X)}{e^x} \right| \leq \frac{\sum_{k=0}^{n} |c_k| x^k}{e^x} \xrightarrow[x\to\infty]{} 0.$$

The logarithm function 1/4

- Since the exponential function $E(x) = e^x$ is strictly increasing and differentiable on $\mathbb{R}$ it has an inverse function L, which is also strictly increasing and differentiable and whose domain is $E[\mathbb{R}] = (0, \infty)$.
- L is defined by
$$E(L(y)) = y \quad \text{for all} \quad y > 0$$
or, equivalently, $L(E(x)) = x$ for all $x \in \mathbb{R}$.
- Differentiating the latter equation
$$1 = (x)' = (L(E(x)))' = L'(E(x))E'(x) = L'(E(x))E(x).$$
Thus $L'(E(x)) = \frac{1}{E(x)}$, hence
$$L'(y) = \frac{1}{y} \quad \text{for all} \quad y > 0.$$

The logarithm function 2/4

- Writing $u = E(x)$ and $v = E(y)$ note that

$$L(uv) = L(E(x)E(y)) = L(E(x+y)) \\ = x + y = L(u) + L(v) \quad \text{for} \quad u, v > 0.$$

- From now on we will write $\log(x) = L(x)$.
- Since $\lim_{x\to+\infty} e^x = +\infty$ and $\lim_{x\to-\infty} e^x = 0$, we conclude

$$\lim_{x\to\infty} \log(x) = +\infty, \quad \text{and} \quad \lim_{x\to 0} \log(x) = \infty.$$

- Observe also that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. By L'Hôpital's rule we have

$$\lim_{n\to\infty} \frac{\log\left(1 + \frac{x}{n}\right)}{\frac{1}{n}} = \lim_{y\to 0} \frac{\log\left(1 + xy\right)}{y} = \lim_{y\to 0} \frac{x}{1 + xy} = x.$$

The logarithm function 3/4. Definition of x^α

- Since $x = E(L(x))$, it is easily seen that

$$x^n = E(nL(x)) \quad \text{and} \quad x^{1/m} = E\left(\frac{1}{m}L(x)\right) \quad \text{for} \quad n, m \in \mathbb{N}.$$

  Thus

$$x^\alpha = E(\alpha L(x)) \quad \text{if} \quad \alpha \in \mathbb{Q}.$$

- It also makes sense to define

$$x^\alpha = E(\alpha L(x)) \quad \text{for} \quad \alpha \in \mathbb{R} \quad \text{and} \quad x > 0.$$

- The continuity and monotonicity of E and L show that everything makes sense and this definition coincides with

$$x^\alpha = \sup\{x^p \ : \ p < \alpha, \ p \in \mathbb{Q}\} \quad \text{if} \quad \alpha \in \mathbb{R} \quad \text{and} \quad x > 1.$$

The logarithm function 4/4

- If we differentiate
$$x^\alpha = E(\alpha L(x)),$$
then
$$(x^\alpha)' = E'(\alpha L(x))\frac{\alpha}{x} = \alpha x^{\alpha-1}.$$
- Finally note that
$$\lim_{x\to\infty} x^{-\alpha}\log(x) = 0 \quad \text{for every} \quad \alpha > 0.$$
That is, $\log(x)$ tends to $+\infty$ slower that any power of x.
- Indeed, since $x^\alpha \underset{x\to\infty}{\longrightarrow} +\infty$, by L'Hôpital's rule
$$\lim_{x\to\infty}\frac{\log(x)}{x^\alpha} \underbrace{=}_{L'Hopital} \lim_{x\to\infty}\frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim_{x\to\infty}\frac{1}{\alpha x^\alpha} = 0.$$

Euler–Mascheroni constant

Divergence of harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Theorem

The sequences

$$a_n = \sum_{k=1}^{n-1} \frac{1}{k} - \log(n) \quad \text{and} \quad b_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n)$$

are increasing and decreasing respectively and bounded, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \gamma.$$

where γ is known as **the Euler (or Euler–Mascheroni) constant**.

Proof: 1/2

Remark

- It is not even known whether γ is irrational.
- γ is called Euler-Mascheroni constant, and $\gamma \simeq 0,5772\ldots$.

Proof. We know

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

thus

$$n\log\left(1+\frac{1}{n}\right) < 1 < (n+1)\log\left(1+\frac{1}{n}\right),$$

and consequently

$$\log\left(\frac{n+1}{n}\right) < \frac{1}{n},$$

$$\log\left(\frac{n+1}{n}\right) > \frac{1}{n+1}.$$

Proof: 2/2

Thus

$$a_{n+1} - a_n = \sum_{k=1}^{n} \frac{1}{k} - \log(n+1) - \sum_{k=1}^{n-1} \frac{1}{k} + \log(n) = \frac{1}{n} - \log\left(\frac{n+1}{n}\right) > 0.$$

Hence $(a_n)_{n\in\mathbb{N}}$ is increasing. Similarly,

$$b_{n+1} - b_n = \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right) < 0,$$

thus $(b_n)_{n\in\mathbb{N}}$ is decreasing. Also it is clear

$$a_1 \leq a_n \leq b_n \leq b_1.$$

Thus by the (MCT) the limits exist

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \gamma,$$

since $b_n = a_n + \frac{1}{n}$. □

Proposition

Proposition

For $x > 0$ one has

$$\frac{x}{x+1} < \frac{2x}{x+2} \leq \log(x+1) < x.$$

Proof. Let $f(x) = x - \log(1+x)$, then

$$f(0) = 0,$$

$$f'(0) = 1 - \frac{1}{x+1} > 0 \quad \Longleftrightarrow \quad x > 0$$

thus f is increasing for $x > 0$. Hence $f(x) > f(0)$ for $x > 0$, so

$$\log(1+x) < x.$$

Proof

We now consider

$$h(x) = \log(1+x) - \frac{2x}{x+1} \quad \text{for} \quad x > 0.$$

Note that $h(0) = 0$ and

$$h'(x) = \frac{x^2}{(x+1)(x+2)^2} > 0 \quad \text{for} \quad x > 0.$$

Thus h is increasing for $x > 0$ and

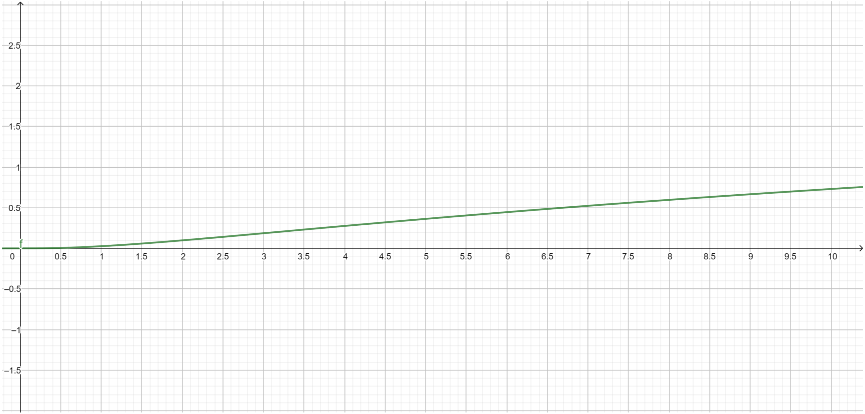
$$h(x) > h(0) = 0.$$

Consequently

$$\log(1+x) > \frac{2x}{x+2} > \frac{x}{x+1}$$

for $x > 0$ as desired. $\square$

Graph of the function $\log(x+1) - \frac{2x}{x+2}$



Application

Application

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}\right)=\log 2.$$

Proof. Note that

$$\frac{1}{n+1}<\log\left(1+\frac{1}{n}\right)<\frac{1}{n}\quad\text{for}\quad n>1$$

upon taking $x=\frac{1}{n}$ in $\frac{x}{x+1}<\log(1+x)<x$. Consequently

$$\log\left(\frac{2n+1}{n}\right)<\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}<\log\left(\frac{2n}{n-1}\right).$$

Thus

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{2n}\right)=\log 2.$$

□

Inequalities between weighted means

Theorem

If $x_1, \ldots, x_k > 0$ and $\alpha_1, \ldots, \alpha_k > 0$ and $\sum_{j=1}^{k} \alpha_j = 1$, then

$$x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k} \leq \alpha_1 x_1 + \ldots + \alpha_k x_k.$$

Proof. Let $f(x) = \log(x)$ and note that

$$f'(x) = \frac{1}{x} \quad \text{and} \quad f''(x) = \frac{-1}{x^2} < 0.$$

Thus $f''(x) < 0$ for all $x > 0$ which means that f is concave. In other words, for all $x_1, \ldots, x_k > 0$ and $\alpha_1, \ldots, \alpha_k > 0$ obeying condition $\alpha_1 + \ldots + \alpha_k = 1$, we have

$$f(\alpha_1 x_1 + \ldots + \alpha_k x_k) \geq \alpha_1 f(x_1) + \ldots + \alpha_k f(x_k).$$

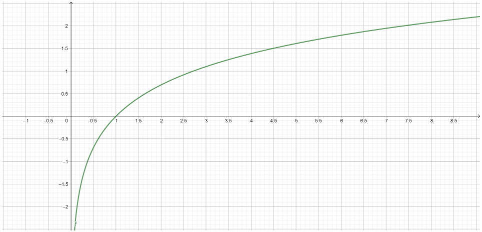
Proof

Consequently, we have

$$\log(x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k}) = \sum_{j=1}^{k} \alpha_j \log(x_j) \leq \log\left(\sum_{j=1}^{k} \alpha_j x_j\right)$$

if and only if

$$x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k} \leq \sum_{j=1}^{k} \alpha_j x_j. \qquad \square$$



Corollary

Corollary

If $p, q > 0$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ and $x, y > 0$, then

$$xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q.$$

Proof. If suffices to apply the previous result with $\alpha_1 = \frac{1}{p}$, $\alpha_2 = \frac{1}{q}$ and $x_1 = x^p$, $x_2 = y^q$, then we obtain

$$xy = x_1^{1/p} x_2^{1/q} \leq \frac{1}{p}x_1 + \frac{1}{q}x_2 = \frac{1}{p}x^p + \frac{1}{q}y^q. \qquad \square$$

Remark

The inequality above is the key in the proof of Hölder's inequality.