Lecture 21

Power series of trigonometric functions done right
Fundamental Theorem of Algebra
and Taylor expansions of other important functions and applications

MATH 411H, FALL 2025

November 13, 2025

Power series

Power series

Given a sequence $(c_n)_{n\in\mathbb{N}_0}$, where $c_n \in \mathbb{R}$, the series

$$\sum_{n=0}^{\infty} c_n x^n, \quad x \in \mathbb{R}$$

is called **a power series**.

- The numbers c_n are called **the coefficients of the series**.

Example 1

$\sum_{n=0}^{\infty} x^n$.

Example 2

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Radius of convergence

Radius of convergence

Given the power series

$$\sum_{n=0}^{\infty} c_n x^n$$

set

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad \text{and} \quad R = \frac{1}{\alpha}.$$

If $\alpha = 0$, then $R = +\infty$.

- The number R is called **the radius of convergence of** $\sum_{n=0}^{\infty} c_n x^n$.

Theorem

Theorem

The series $\sum_{n=0}^{\infty} c_n x^n$

- converges if $|x| < R$, and
- diverges if $|x| > R$.

Proof. Consider $a_n = c_n x^n$ and apply the root test

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |x| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|x|}{R}. \qquad \square$$

Example 1

$\sum_{n=0}^{\infty} n^n x^n$ has $R = 0$

Example 2

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has $R = +\infty$, since $\alpha = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0$.

Examples

Example 3

$\sum_{n=0}^{\infty} x^n$ has $R = 1$. If $|x| = 1$ the series $\sum_{n=0}^{\infty} x^n$ diverges. We also know

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad |x| < 1.$$

Example 4

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ has $R = 1$. If $x = 1$ the series diverges since $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$. If $x = -1$ then the series converges since

$$\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right| < \infty.$$

C and S functions

Recall that $i^2 = -1$. Let us define

$$C(x) = \frac{1}{2} \cdot (E(ix) + E(-ix)), \quad \text{and}$$
$$S(x) = \frac{1}{2i} \cdot (E(ix) - E(-ix))$$

- We shall show that $C(x)$ and $S(x)$ coincide with the functions $\cos(x)$ and $\sin(x)$, whose definition is usually based on geometric considerations.
- It is easy to see that $E(\overline{z}) = \overline{E(z)}$, since

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

- Hence, we have $\overline{C(x)} = C(x)$ and $\overline{S(x)} = S(x)$, so $C(x)$ and $S(x)$ are real for $x \in \mathbb{R}$. Recall that a number $z \in \mathbb{C}$ is real if $z = \overline{z}$, and the outputs of the functions $C(x)$ and $S(x)$ satisfy this relation.

Euler's formula

- Also Euler's formula holds

$$E(ix) = C(x) + iS(x) \quad \text{if} \quad x \in \mathbb{R}.$$

- Moreover, $|E(ix)| = 1$ for any $x \in \mathbb{R}$, since

$$|E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = 1.$$

- Since

$$C(x) = \frac{1}{2}(E(ix) + E(-ix)) \quad \text{and} \quad S(x) = \frac{1}{2i}(E(ix) - E(-ix))$$

we can read off that $C(0) = 1$ and $S(0) = 0$ and also

$$C'(x) = -S(x) \quad \text{and} \quad S'(x) = C(x)$$

Since

$$C'(x) = \frac{1}{2}(iE(ix) - iE(-ix)) = \frac{i}{2}(E(ix) - E(-ix)) = -S(x).$$

Zeroes of the function C and definition of π

We assert that there exist positive numbers x such that $C(x) = 0$.

- If not, since $C(0) = 1$, it follows that $C(x) > 0$ for all $x > 0$. If $C(x_1) < 0$ for some $x_1 > 0$, then by the intermediate value theorem, since C is continuous, $C(x_2) = 0$ for some $0 < x_2 < x_1$, contradiction!
- Hence $S'(x) > 0$ since
$$S'(x) = C(x).$$
But $S(0) = 0$, thus $S(x)$ is strictly increasing on $(0, \infty)$.
- By the mean–value theorem
$$C(y) - C(x) = -(y - x) \cdot S(\theta_{x,y}) \quad \text{for some} \quad \theta_{x,y} \in (x, y)$$
thus
$$C(x) - C(y) = (y - x) \cdot S(\theta_{x,y}) > (y - x)S(x)$$
and since $|C(x)| \leq 1$ and $|S(x)| \leq 1$ we conclude
$$(y - x)S(x) \leq C(x) - C(y) \leq 2.$$
- But this is impossible if y is large since $S(x) > 0$.

Zeroes of the function C and definition of π

- Let $x_0 > 0$ be the smallest number such that $C(x_0) = 0$. This exists since $C(0) = 1$ and the set of zeroes is closed.
- We define the number π to be
$$\pi = 2x_0.$$
- Then $C(\frac{\pi}{2}) = 0$ and since $|E(ix)| = 1$ we deduce
$$S\left(\frac{\pi}{2}\right) = \pm 1.$$
- Since $C(x) > 0$ in $(0, \frac{\pi}{2})$, S is increasing in $(0, \frac{\pi}{2})$. Hence $S(\frac{\pi}{2}) = 1$.
- Thus we have
$$E\left(\frac{\pi i}{2}\right) = i.$$
- Since $E(z + w) = E(z)E(w)$ thus we have
$$E(\pi i) = -1 \quad \text{and} \quad E(2\pi i) = 1,$$
hence $E(z + 2\pi i) = E(z)$ for all $z \in \mathbb{C}$.

Theorem

Theorem

(i) *The exponential function* $E : \mathbb{C} \to \mathbb{C}$ *is periodic with period* $2\pi i$.

(ii) *The functions* S *and* C *are periodic with period* 2π.

(iii) *If* $0 < t < 2\pi$, *then* $E(it) \neq 1$.

(iv) *If* $z \in \mathbb{C}$ *and* $|z| = 1$, *there is a unique* $t \in (0, 2\pi)$ *such that* $E(it) = z$

Proof. Item (i) easily follows since $E(z + 2\pi i) = E(z)$ for all $z \in \mathbb{C}$. Since $C(x) = \frac{1}{2}(E(ix) + E(-ix))$ we see

$$\begin{aligned} C(x + 2\pi) &= \frac{1}{2}(E(i(x + 2\pi)) + E(-i(x + 2\pi))) \\ &= \frac{1}{2}(E(ix) + E(-ix)) = C(x). \end{aligned}$$

Similarly, $S(x) = S(x + 2\pi)$. Thus (ii) is proved.

Proof

- To prove (c) suppose $0 < t < \frac{\pi}{2}$ and

$$E(it) = x + iy \quad \text{with} \quad x, y \in \mathbb{R}.$$

- Our preceding discussion shows that $0 < x < 1$ and $0 < y < 1$.
- Now note that $0 < t < \frac{\pi}{2} \iff 0 < 4t < 2\pi$ and that

$$E(4it) = E(it)^4 = (x + iy)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2)$$

- If $E(4it) \in \mathbb{R}$, then it follows that $x^2 - y^2 = 0$. We are only considering real $E(4it)$ because if $E(4it)$ is imaginary, then it clearly is not equal to 1. Since

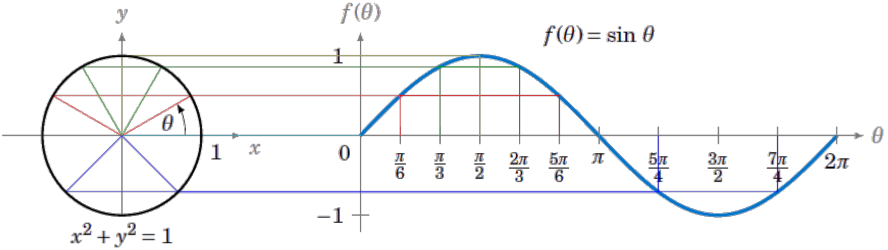
$$|E(it)| = 1 = x^2 + y^2,$$

so we have $x^2 = y^2 = \frac{1}{2}$. Hence $E(4it) = -1$ and we are done.

Proof

- To prove (iv), if $0 \leq t_1 < t_2 < 2\pi$, then uniqueness follows, since
$$E(it_2)E(it_1)^{-1} = E(i(t_2 - t_1)) \neq 1.$$
- To prove the existence we fix $z \in \mathbb{C}$ so that $|z| = 1$. Write $z = x + iy$ with $x, y \in \mathbb{R}$. Suppose first that $x \geq 0$ and $y \geq 0$.
- $C(t)$ decreases on $[0, \frac{\pi}{2}]$ from 1 to 0. Hence $C(t) = x$ for some $t \in [0, \frac{\pi}{2}]$. Since $C^2 + S^2 = 1$ and $S \geq 0$ on $[0, \frac{\pi}{2}]$, then $z = E(it)$.
- If $x < 0$ and $y \geq 0$, the preceding conditions are satisfied by $-iz$. Hence $-iz = E(it)$ for some $t \in [0, \frac{\pi}{2}]$.
- Since $i = E\left(\frac{\pi i}{2}\right)$ we obtain
$$z = E\left(i\left(t + \frac{\pi}{2}\right)\right).$$
- If $y < 0$, the preceding two cases show that $z = E(it)$ for some $t \in (0, \pi)$. Thus, $z = -E(it) = E(i(t + \pi))$ since $E(i\pi) = -1$. $\square$

sin and cos functions



Considerations of the triangle whose vertices are

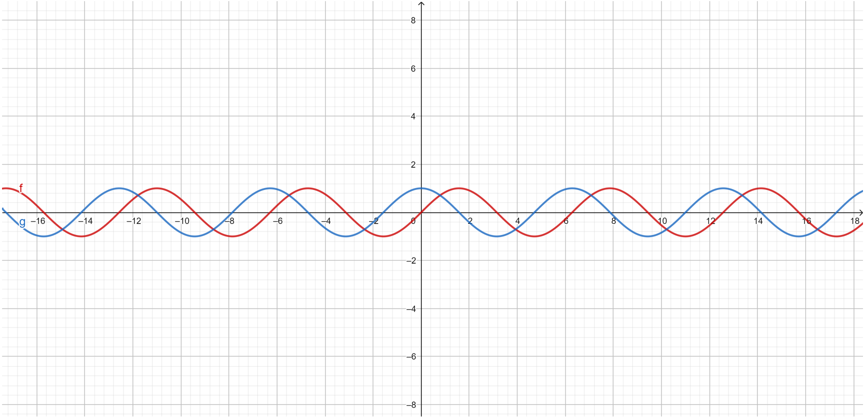
$$z_1 = 0, \qquad z_2 = \gamma(\theta), \qquad z_3 = C(\theta)$$

show that $\cos(\theta) = C(\theta)$ and $\sin(\theta) = S(\theta)$.

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \qquad \text{and} \qquad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.$$

Graphs of sin(x) and cos(x)

Now we can sketch the graphs of sin(x) and cos(x):



Remark

Remark

- The curve $\gamma(t) = E(it)$, with $(0 \leq t \leq 2\pi$, is a simple closed curve whose range is the unit circle in the plane.
- Since $\gamma'(t) = iE(it)$, the length of γ as we shall see soon is

$$\int_0^{2\pi} |\gamma'(t)| dt = 2\pi.$$

- This is of course the expected result of the circumference of a circle of radius 1.
- In the same way we see that the point $\gamma(t)$ describes a circular arc length t_0 as t increases from 0 to t_0.

Limit $\lim_{h\to 0} \frac{\sin(h)}{h}$

Theorem

We have

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1.$$

Proof. We have

$$\lim_{h\to 0} \frac{\sin(h)}{h} = \lim_{h\to 0} \frac{\sin(h) - \sin(0)}{h} = \sin'(0) = \cos(0) = 1. \qquad \square$$

Taylor's theorem

Taylor's theorem

Suppose $f : [a, b] \to \mathbb{R}$, $n \in \mathbb{N}$, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let $\alpha, \beta \in [a, b]$ be distinct and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

then there exists a point $x \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)(x)}}{n!}(\beta - \alpha)^n \qquad (*)$$

Proof 1/3

Remark

For $n = 1$ this is just **the mean–value theorem**. In general, the theorem says that f can be approximated by a polynomial of degree $n - 1$ and that (*) allows us to estimate the error term if we know bounds on $|f^{(n)}(x)|$.

Proof. Let M be a number such that

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n.$$

For $a \leq t \leq b$ set

$$g(t) = f(t) - P(t) - M(t - \alpha)^n.$$

- We have to show $n!M = f^{(n)}(x)$ for some $x \in (\alpha, \beta)$.

Proof 2/3

- Since
$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$
we have that $P^{(n)}(t) = 0$. Thus
$$g^{(n)}(t) = f^{(n)}(t) - n!M \quad \text{for} \quad t \in (\alpha, \beta)$$
(since $(x^n)^{(n)} = n!$).
- The proof will be completed if we show that $g^{(n)}(x) = 0$ for some $x \in (\alpha, \beta)$. Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, 1, 2, \ldots, n-1$, hence we have
$$g(\alpha) = g'(\alpha) = \ldots = g^{(n-1)}(\alpha) = 0.$$
Our choice of M shows that $g(\beta) = 0$.

Proof 3/3

- Hence by **the mean-value theorem**

$$g'(x_1) = 0 \quad \text{for some} \quad x_1 \in (\alpha, \beta)$$

since $0 = g(\alpha) - g(\beta) = (\beta - \alpha)g'(x_1)$.

- Using that $g'(\alpha) = 0$ we continue and obtain

$$0 = g'(x_1) - g'(\alpha) = (x_1 - \alpha)g''(x_2) \quad \text{for some} \quad \alpha < x_2 < x_1.$$

Thus $g''(x_2) = 0$.

- Repeating the previous arguments, after n steps we obtain

$$g^{(n)}(x_n) = 0 \quad \text{for some} \quad \alpha < x_n < x_{n-1} < \ldots < x_1 < \beta.$$

This completes the proof. □

Theorem

Theorem (Taylor's expansion formula)

Suppose that $f : [a, b] \to \mathbb{R}$ is n-times continuously differentiable on $[a, b]$ and $f^{(n+1)}$ exists in the open interval (a, b). For any $x, x_0 \in [a, b]$ and $p > 0$ there exists $\theta \in (0, 1)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r_n(x),$$

where $r_n(x)$ is **the Schlömlich–Roche remainder** function defined by

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!p}(1 - \theta)^{n+1-p}(x - x_0)^{n+1}.$$

Proof 1/3

- For $x, x_0 \in [a, b]$ set

$$r_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

- Wlog we may assume that $x > x_0$. For $z \in [x_0, x]$ define

$$\phi(z) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x - z)^k.$$

- We have $\phi(x_0) = r_n(x)$ and $\phi(x) = 0$, and ϕ' exists in (x_0, x) and

$$\phi'(z) = -\frac{f^{(n+1)}(z)}{n!}(x - z)^n.$$

Proof 2/3

- Indeed, by the telescoping we obtain

$$\begin{aligned}
\phi'(z) &= -\left(\sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!}(x-z)^k\right)' \\
&= -\sum_{k=0}^{n}\left(\frac{f^{(k+1)}(z)}{k!}(x-z)^k - \frac{f^{(k)}(z)}{k!}k(x-z)^{k-1}\right) \\
&= \sum_{k=1}^{n} \frac{f^{(k)}(z)}{(k-1)!}(x-z)^{k-1} - \sum_{k=0}^{n} \frac{f^{(k+1)}(z)}{k!}(x-z)^k \\
&= -\frac{f^{(n+1)}(z)}{n!}(x-z)^n.
\end{aligned}$$

- Let $\psi(z) = (x-z)^p$, then ψ is continuous on $[x_0, x]$ with non-vanishing derivative on (x_0, x).

Proof 3/3

- By **the mean-value theorem**

$$\frac{\phi(x)-\phi(x_0)}{\psi(x)-\psi(x_0)} = \frac{\phi'(c)}{\psi'(c)} \quad \text{for some} \quad c \in (x_0, x).$$

- Thus, setting $c = x_0 + \theta(x - x_0)$,

$$\begin{aligned} r_n(x) &= \underbrace{\phi(x_0)}_{=r_n(x)} - \underbrace{\phi(x)}_{=0} = -(\psi(x)-\psi(x_0))\frac{\phi'(c)}{\psi'(c)} \\ &= \frac{f^{(n+1)}(c)}{n!}(x-c)^n \frac{-(x-x_0)^p}{-p(x-c)^{p-1}} = \\ &= \frac{f^{(n+1)}(x_0+\theta(x-x_0))}{pn!}(1-\theta)^{n+1-p}(x-x_0)^{n+1}. \quad \square \end{aligned}$$

Corollary

Under the assumptions of the previous theorem.

Lagrange remainder

If $p = n + 1$ we obtain the Taylor formula with **the Lagrange remainder**:

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{(n+1)!}(x - x_0)^{n+1}.$$

Cauchy remainder

If $p = 1$ we obtain the Taylor formula with **the Cauchy remainder**:

$$r_n(x) = \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!}(1 - \theta)^n (x - x_0)^{n+1}.$$

Power series expansion for the logarithm

Theorem

For $|x| < 1$ we have

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k.$$

Proof. Note that $(\log(x+1))' = \frac{1}{x+1}$ and

$$(\log(x+1))'' = \left(\frac{1}{x+1}\right)' = -\frac{1}{(1+x)^2},$$

$$(\log(x+1))''' = \left(= -\frac{1}{(1+x)^2}\right) = \frac{2}{(1+x)^3},$$

$$(\log(x+1))^{(4)} = \left(\frac{2}{(1+x)^3}\right)' = -\frac{6}{(1+x)^4} = -\frac{3!}{(1+x)^4}.$$

Proof 1/2

Inductively, we have

$$(\log(1+x))^{(n)} = (-1)^{n+1}\frac{(n-1)!}{(x+1)^n}.$$

- We use the Taylor expansion formula at $x_0 = 0$ then

$$\log(1+x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + r_n(x) = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k}x^k + r_n(x),$$

since

$$f^{(0)}(0) = \log(1) = 0,$$
$$f^{(k)}(0) = (-1)^{k+1}(k-1)!.$$

Proof 2/2

- If $0 \leq x < 1$ we use Lagrange's reminder. Then for some $0 < \theta < 1$,

$$|r_n(x)| = \left| \frac{f^{(n)}(\theta x)}{(n+1)!} x^{n+1} \right| = \frac{n!}{(n+1)!(1+\theta x)^n} x^{n+1} \leq \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0.$$

- If $-1 < x < 0$ we use Cauchy's remainder. Then for some $0 < \theta < 1$,

$$|r_n(x)| = \left| \frac{f^{(n+1)}(x_0 + \theta(x - x_0))}{n!} (1-\theta)^n (x - x_0)^{n+1} \right|$$
$$= \left| \frac{n!}{n!(1+\theta x)^{n+1}} (1-\theta)^n x^{n+1} \right|.$$

- Since $-1 < \theta x < 0$, then $-\theta < \theta x$, so $1 - \theta < 1 + \theta x$, hence

$$|r_n(x)| \leq \frac{(1-\theta)^n}{(1+\theta x)^{n+1}} |x|^{n+1} \leq \frac{(1-\theta)^n}{(1-\theta)^{n+1}} |x|^{n+1} = \frac{|x|^{n+1}}{1-\theta} \xrightarrow[n\to\infty]{} 0$$

since $|x|^n \xrightarrow[n\to\infty]{} 0$ when $|x| < 1$. □

Newton's binomial formula

Theorem

If $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and $|x| < 1$ then

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \underbrace{\frac{\alpha(\alpha-1)\cdot \ldots \cdot (\alpha-n+1)}{n!}}_{\binom{\alpha}{n}} x^n.$$

This is called **Newton's binomial formula**.

Recall

For $n \in \mathbb{N}$ we have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdot \ldots \cdot (n-k+1)}{k!}.$$

Proof 1/4

Proof. Let let $f(x) = (1+x)^\alpha$ and note that

$$f^{(n)}(x) = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)x^{\alpha-n}.$$

- Suppose first that $0 < x < 1$.
  Using the Lagrange remainder formula we have

$$r_n(x) = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}(1+x\theta)^{\alpha-n+1}.$$

Claim

For $|x| < 1$ we have

$$\lim_{n\to\infty}\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1} = 0.$$

Proof 2/4. Proof of the Claim.

- To prove the claim it suffices to use the following fact:

Fact

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = q < 1 \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = 0$$

with $a_n = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n-1)x^{n+2}}{(n+2)!}\frac{(n+1)!}{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)x^{n+1}}\right|$$

$$= \left|\frac{\alpha-n-1}{n+2}x\right| \xrightarrow[n\to\infty]{} |x| < 1.$$

- Thus $r_n(x) \xrightarrow[n\to\infty]{} 0$ if we show that $(1+\theta x)^{\alpha-n-1}$ is bounded.

Proof 3/4

- Indeed, assuming that $0 < x < 1$ we see

$$(1+\theta x)^{-n} \leq 1,$$

- For $\alpha \geq 0$ we have

$$1 \leq (1+\theta x)^{\alpha} \leq (1+x)^{\alpha} \leq 2^{\alpha},$$

- For $\alpha < 0$ we have

$$2^{\alpha} \leq (1+x)^{\alpha} \leq (1+x\theta)^{\alpha} \leq 1$$

- Gathering all together we conclude that $(1+\theta x)^{\alpha-n-1}$ as desired.

Proof 4/4

- Now we assume that $-1 < x < 0$. Using the Cauchy remainder formula we have

$$r_n(x) = \frac{\alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n)}{(n+1)!}x^{n+1}(1-\theta)^n(1+\theta x)^{\alpha-n-1}.$$

As before we show that $(1-\theta)(1+\theta x)^{\alpha-n-1}$ is bounded.

- Since $-1 < x < 0$ then $1+\theta x > 1-\theta$ and consequently

$$(1-\theta)^n \leq (1-\theta)^n(1+\theta x)^{-n} = \frac{(1-\theta)^n}{(1+\theta x)^n} < 1.$$

- For $\alpha \leq 1$ we have

$$1 \leq (1+x\theta)^{\alpha-1} \leq (1+x)^{\alpha-1}.$$

- For $\alpha \geq 1$ we have

$$(1+x)^{\alpha-1} \leq (1+\theta x)^{\alpha-1} \leq 1$$

and we are done. □

A function which does not have power series representation

Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- It is not difficult to see that f is infinitely many times differentiable for any $x \in \mathbb{R}$.
- Moreover,

$$f^{(n)}(0) = 0 \quad \text{for any} \quad n \geq 0$$

  and $f(x) \neq 0$.
- Thus we see

$$0 \neq f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0.$$

Bernoulli's inequality: general form

Bernoulli's inequality: general form

For $x > -1$ and $x \neq 0$ we have

a. $(1+x)^\alpha > 1 + \alpha x$ if $\alpha > 1$ or $\alpha < 0$,
b. $(1+x)^\alpha < 1 + \alpha x$ if $0 < \alpha < 1$.

Proof. Applying Taylor's formula with the Lagrange remainder for $f(x) = (1+x)^\alpha$ we obtain

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} x^2.$$

Proof

- For $\alpha > 1$ or $\alpha < 0$ we have
$$\frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} > 0.$$
- For $0 < \alpha < 1$ we have
$$\frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} < 0.$$
- Consequently, for $\alpha > 1$ or $\alpha < 0$ we obtain
$$1 + \alpha x + \frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} x^2 > 1 + x\alpha.$$
- Similarly, for $0 < \alpha < 1$, we obtain
$$1 + \alpha x + \frac{\alpha(\alpha-1)(1+\theta x)^{\alpha-2}}{2} x^2 > 1 + x\alpha.$$

This completes the proof. □