Lecture 22

Riemann Integrals

MATH 411H FALL 2025

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Partitions

Partition

Let $[a, b]$ be a given interval. By a partition P of $[a, b]$ we mean a finite set of points

$$a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b.$$

Example 1

If $[a, b] = [0, 1]$, then $\{0, \frac{1}{2}, 1\}$ is a partition.

Example 2

If $[a, b] = [0, 1]$, then

$$\left\{ \frac{k}{n} \ : \ k = 0, 1, \ldots, n \right\}$$

is a partition for every $n \in \mathbb{N}$.

Upper and lower Riemann sums

Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Corresponding to each partition P of $[a, b]$ we put

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad \text{and} \quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x),$$

$$\Delta x_i = x_i - x_{i-1}.$$

Upper and lower Riemann sums

We define

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

- We always have that $L(P, f) \leq U(P, f)$.

Examples

Example 1

If $f(x) = x$ and $P = \{0, \frac{1}{2}, 1\}$, then

$$U(P, f) = \frac{1}{2} \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{3}{4}, \quad \text{and} \quad L(f, L) = 0 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Example 2

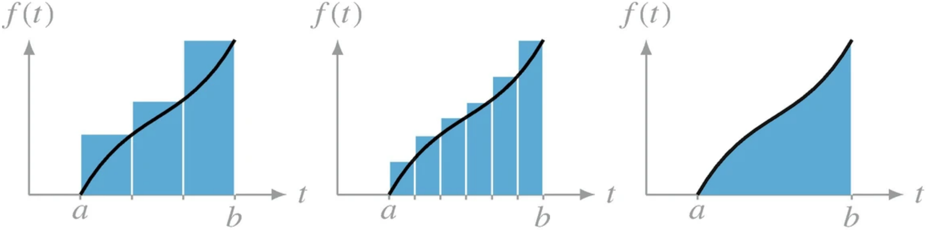
If $f(x) = x^2$ and

$$P = \left\{ \frac{k}{n} \ : \ k = 0, 1, \ldots, n \right\},$$

then

$$U(P, f) = \sum_{i=1}^{n} \frac{1}{n} \left(\frac{i}{n} \right)^2, \quad \text{and} \quad L(P, f) = \sum_{i=1}^{n} \frac{1}{n} \left(\frac{i-1}{n} \right)^2.$$

Riemann sums - geometric interpretation



Upper and lower Riemann integral

Upper and lower Riemann integral

We define the **upper and lower Riemann integrals of f over** $[a, b]$ to be

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P, f), \quad \text{and} \quad \overline{\int_a^b} f(x)\,dx = \inf_P U(P, f),$$

where the inf and the sup are taken over all partitions P of $[a, b]$.

Riemann integral of f over $[a, b]$

If the upper and lower integrals are equal, we say that $f : [a, b] \to \mathbb{R}$ is **Riemann integrable** on $[a, b]$, we write $f \in \mathcal{R}([a, b])$ and we denote the common value (which is called Riemann integral of f over $[a, b]$) by

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx.$$

Riemann integral is well-defined

Fact

The upper and lower integrals are defined for every bounded function.

Proof. Let

$$m = \inf_{x\in[a,b]} f(x),$$
$$M = \sup_{x\in[a,b]} f(x).$$

Then

$$m \le f(x) \le M \quad \text{for all} \quad x \in [a,b].$$

Therefore

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

for every partition P. □

Question of the integrability of f

Example

There is a bounded function f which is not integrable.

Proof. Let us define f on $[0, 1]$ to be

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q}, \\ 0 \text{ if } x \notin \mathbb{Q}. \end{cases}$$

Let us recall

Fact (*)

In any interval $[c, d]$ such that $c < d$ there is a rational and irrational number.

Proof

By the fact $(*)$, for every partition P of $[0,1]$, we have

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1,$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0.$$

Therefore

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P,f) = 0, \quad \text{and} \quad \overline{\int_a^b} f(x)\,dx = \inf_P U(P,f) = 1.$$

Hence $\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$ and f is not integrable. □

Refinements

Refinement

We say that the partition P^* is a refinement of P if $P^* \supseteq P$.

Common refinement

Given two partitions, P_1 and P_2,we say that P^* is their common refinement if

$$P^* = P_1 \cup P_2.$$

Example

If $[a, b] = [0, 2]$ and $P_1 = \{0, \frac{1}{2}, 1, 2\}$, $P_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{2}, 2\}$ are partitions, then their common refinement is

$$P^* = \left\{0, \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2\right\}.$$

Theorem

Theorem

If P^* is a refinement of P, then

$$L(P, f) \le L(P^*, f),$$
$$U(P^*, f) \le U(P, f).$$

Proof. We prove the first statement.

- Suppose first that P^* contains just one point more than P. Let this extra point be x^*, and suppose

$$x_{i-1} \le x^* \le x_i \quad \text{for some} \quad i \in \{1, 2, \dots, n\}.$$

- Let

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

$$w_1 = \inf_{x \in [x_{i-1}, x^*]} f(x), \quad \text{and} \quad w_2 = \inf_{x \in [x^*, x_i]} f(x)$$

Proof

- Then $w_1 \geq m_i$ and $w_2 \geq m_i$ and consequently

$$L(P^*, f) - L(P, f) = w_1(x^* - x_{i-1}) + w_2(x_i - x^*) - m_i(x_i - x_{i-1})$$
$$= (w_1 - m_i)(x^* - x_{i-1}) + (w_2 - m_i)(x_i - x^*) \geq 0.$$

- Finally, if P^* contains k points more than P, we repeat this reasoning k times. The proof of the second statement is analogous. □

Claim (*)

For two partitions P_1, P_2 of an interval $[a, b]$ one has

$$L(P_1, f) \leq U(P_2, f).$$

Proof. Let $P^* = P_1 \cup P_2$ be the common refinement of two partitions P_1 and P_2. By the previous theorem

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f).$$

Theorem

Theorem

For any bounded function $f : [a, b] \to \mathbb{R}$ we have

$$\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx.$$

Proof. By the Claim (*) for two partitions P_1, P_2 of an interval $[a, b]$ one has

$$L(P_1, f) \leq U(P_2, f).$$

Then

$$\underline{\int_a^b} f(x)\, dx = \sup_{P_1} L(P_1, f) \leq \inf_{P_2} U(P_2, f) = \overline{\int_a^b} f(x)\, dx,$$

This completes the proof of the theorem. □

Theorem

A function $f \in \mathcal{R}([a,b])$ if and only if the following condition $(\mathcal{R})$ holds:

- For every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon. \qquad (\mathcal{R})$$

Proof. By the previous theorem, for every partition P we have

$$L(P,f) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq U(P,f).$$

Thus the condition $(\mathcal{R})$ implies

$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(P,f) - L(P,f) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, hence $f \in \mathcal{R}(\alpha)$.

Proof

Conversely, suppose that $f \in \mathcal{R}(\alpha)$. Then for every $\varepsilon > 0$ there are partitions P_1 and P_2 such that

$$\underline{\int_a^b} f(x)dx - L(P_1, f) < \frac{\varepsilon}{2} \quad \text{and} \quad U(P_2, f) - \overline{\int_a^b} f(x)dx < \frac{\varepsilon}{2}.$$

We choose P to be the **common refinement** of P_1 and P_2. Then

$$U(P, f) \le U(P_2, f)$$
$$\le \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2} = \int_a^b f(x)dx + \frac{\varepsilon}{2} = \underline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$$
$$\le L(P_1, f) + \varepsilon \le L(P, f) + \varepsilon.$$

This proves condition $(\mathcal{R})$ and completes the proof of the theorem. □

Theorem (**)

If condition $(\mathcal{R})$ holds for $P = \{x_0, \ldots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i < \varepsilon.$$

Proof. Note that $f(s_i), f(t_i)$ lies in $[m_i, M_i]$, hence by the triangle inequality

$$|f(t_i) - f(s_i)| \leq \underbrace{M_i - m_i}_{\text{length}}.$$

Hence

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta x_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta x_i = \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon.$$

This completes the proof. □

Theorem

Theorem

If $f \in \mathcal{R}([a,b])$ and the hypotheses of $(**)$ hold, then

$$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\,dx\right| < \varepsilon.$$

Proof. It is enough to note that

$$L(P,f) \le \sum_{i=1}^{n} f(t_i)\Delta x_i \le U(P,f),$$

$$L(P,f) \le \int_a^b f(x)\,dx \le U(P,f). \qquad \square$$

Riemann integrability for continuous functions

Theorem

If f is continuous on $[a, b]$ then $f \in \mathcal{R}([a, b])$.

Proof. Let $\varepsilon > 0$ be given. Choose $\eta > 0$ such that $\eta(b - a) < \varepsilon$. Recall that **if f is continuous on $[a, b]$, then it is also uniformly continuous.**

- Hence, there is $\delta > 0$ such that $|f(x) - f(t)| < \eta$ if $|t - x| < \delta$.
- In particular, that means that $M_i - m_i < \eta$ for every partition such that $\Delta x_i < \delta$.
- Hence,

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq \eta \sum_{i=1}^{n} \Delta x_i = \eta(b - a) < \varepsilon.$$

The proof is completed. □

Riemann integrability for monotonic functions

Theorem

If $f : [a, b] \to \mathbb{R}$ is monotonic, then $f \in \mathcal{R}([a, b])$.

Proof. Let $\varepsilon > 0$ be given. For $n \in \mathbb{N}$ choose a partition P such that $\Delta x_i = \frac{b-a}{n}$. We suppose that f is monotonically increasing. Then

$$M_i - m_i = f(x_i) - f(x_{i-1}).$$

Hence, if n is taken large enough, we obtain

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \\ &= \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = \frac{b-a}{n}(f(b) - f(a)) < \varepsilon, \end{aligned}$$

and we are done, the proof is analogous in the other case. □

Example

Example 1

Let

$$f(x) = \begin{cases} x & \text{for } x \in [0,1], \\ x^2 + 5 & \text{for } x \in (1,2], \\ x^3 + 9 & \text{for } x \in (3,4]. \end{cases}$$

Prove that f is Riemann integrable on $[0,4]$.

Solution. f is increasing, so f is Riemann integrable by the previous theorem.

Riemann integrability for discontinues functions

Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is bounded and has only finitely many points of discontinuity on $[a, b]$. Then $f \in \mathcal{R}([a, b])$.

Proof. Let $\varepsilon > 0$ be given. Put $M = \sup_{x \in [a,b]} |f(x)|$, let E be the set of points at which f is discontinuous.

- Since E is finite, E can be covered by finitely many disjoint intervals $[u_i, v_i] \subseteq [a, b]$ such that $v_i - u_i < \varepsilon$.
- Furthermore, we can place these intervals in such a way that every point of $E \cap (a, b)$ lies in the interior of some $[u_i, v_i]$.
- Remove the segments (u_i, v_i) from $[a, b]$.
- The remaining set K is **compact**. Hence f is uniformly continuous on K, and there exists $\delta > 0$ such that

$$|f(s) - f(t)| < \varepsilon \quad \text{if} \quad s, t \in K \quad \text{and} \quad |s - t| < \delta.$$

Proof - construction of a partition

- Now form a partition P of $[a, b]$ as follows:
  1. Each u_i occurs in P.
  2. Each v_i occurs in P.
  3. No point of any segment (u_i, v_i) occurs in P.
  4. If x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.
- Note that $M_i - m_i \leq 2M$ for all i, and by the uniform continuity of f we also have that $M_i - m_i \leq \varepsilon$ unless x_{i-1} is one of the u_j.
- Hence

$$U(P, f) - L(P, f) \leq \varepsilon(b - a) + 2M\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is finished. □

Exercise

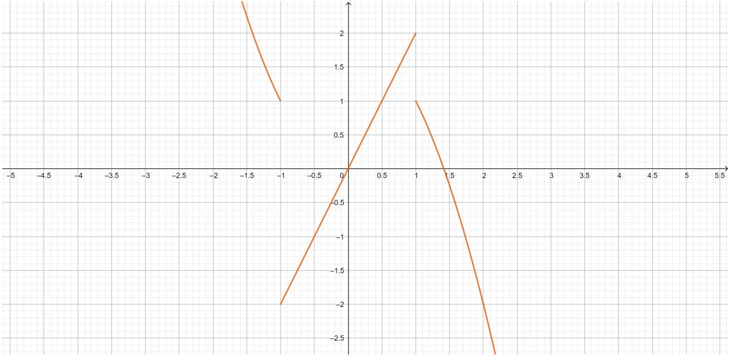
Exercise

Prove that the function given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in [-3,-1], \\ 2x & \text{if } x \in (-1,1], \\ -x^2+2 & \text{if } x \in (1,3] \end{cases}$$

is Riemann integrable.

Exercise - solution



- The function is continuous except -1, 1. Hence, by the previous theorem, it is Riemann integrable.

Linearity of Riemann integrals

Linearity of Riemann integrals

If $f_1, f_2 \in \mathcal{R}([a,b])$ and $c \in \mathbb{R}$, then $f_1 + f_2 \in \mathcal{R}([a,b])$ and

$$\int_a^b f_1(x) + f_2(x)\, dx = \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx,$$

$$\int_a^b c f_1(x)(x)\, dx = c \int_a^b f_1(x)\, dx.$$

Proof. We only prove the first statement. The proof of the second statement is similar.

- If $f = f_1 + f_2$, then for any partition P we have

$$L(P, f_1) + L(P, f_2) \le L(P, f) \le U(P, f) \le U(P, f_1) + U(P, f_2). \quad (*)$$

- For a given $\varepsilon > 0$ there are partitions P_1, P_2 such that $U(P_j, f_j) - L(P_j, f_j) < \varepsilon$ for $j = 1, 2$.

Proof

- Let P be the common refinement of P_1 and P_2. Together with $(*)$ this means that $U(P, f) - L(P, f) < 2\varepsilon$, which proves $f \in \mathcal{R}([a, b])$.
- With the same P we have

$$U(P, f_j) \leq \int_a^b f_j(x)\, dx + \varepsilon$$

- Hence $(*)$ implies

$$\int_a^b f(x)\, dx \leq U(P, f) \leq \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx + 2\varepsilon.$$

- Since $\varepsilon > 0$ we arbitrary, we have

$$\int_a^b f(x)\, d\alpha(x) \leq \int_a^b f_1(x)\, d\alpha(x) + \int_a^b f_2(x)\, d\alpha(x).$$

- If we replace f_1 and f_2 with $-f_1$ and $-f_2$ respectively, the inequality can be reversed, which proves the desired equality. □

Properties of Riemann integrals

Properties of Riemann–Stieltjes integral

Assume $f_1, f_2 \in \mathcal{R}([a,b])$.

1. If $f_1 \le f_2$, then
$$\int_a^b f_1(x)\,dx \le \int_a^b f_2(x)\,dx.$$
2. If $a < c < b$ and $f_1 \in \mathcal{R}([a,c])$ and $f_1 \in \mathcal{R}([c,b])$, then
$$\int_a^b f_1(x)\;dx = \int_a^c f_1(x)\;dx + \int_c^b f(x)\;dx.$$

Composition and product of Riemann integrable functions

Theorem (*)

Suppose $f \in \mathcal{R}([a, b])$, and $m \leq f \leq M$, and $\phi : [m, M] \to \mathbb{R}$ and

$$h(x) = \phi(f(x)).$$

Then $h \in \mathcal{R}([a, b])$. Prove it!

As a consequence of this theorem, we have the following facts:

Fact

Assume that $f, g \in \mathcal{R}([a, b])$. Then

- (a) $fg \in \mathcal{R}([a, b])$,
- (b) $\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$.

Proof

- For the proof of (a), it is enough to use the following identity

$$4fg = (f+g)^2 - (f-g)^2,$$

  the fact that $f+g$, $f-g$ are integrable and Theorem $(*)$ with $\phi(t) = t^2$.

- For the proof of (b), choose $c \in \mathbb{R}$ such that

$$\left|\int_a^b f(x)\,dx\right| = c\int_a^b f(x)\,dx = \int_a^b cf(x)\,dx > 0.$$

  then by Theorem $(*)$ with $\phi(t) = |t|$ we obtain

$$\left|\int_a^b f(x)\,dx\right| = c\int_a^b f(x)\,dx = \int_a^b cf(x)\,dx \le \int_a^b |f(x)|\,dx. \qquad \square$$

Change of variable formula

Theorem (change of variable formula)

Suppose that ϕ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$ and $\phi' \in \mathcal{R}([A, B])$. Suppose that $f \in \mathcal{R}([a, b])$, then $f \circ \phi \in \mathcal{R}([A, B])$ and

$$\int_a^b f(x)\,dx = \int_A^B f(\phi(x))\phi'(x)\,dx.$$

Proof. To each partition $Q = \{y_0, \ldots, y_n\}$ of $[a, b]$ corresponds a partition $P = \{x_0, \ldots, x_n\}$ of $[A, B]$, so that

$$y_i = \phi(x_i).$$

All partitions of $[A, B]$ are obtained in this way.

Proof 1/2

- Let $\varepsilon > 0$. There is a partition P such that $U(P, \phi') - L(P, \phi') < \varepsilon$. The mean-value theorem furnishes points $t_i \in (x_{i-1}, x_i)$ such that

$$y_i - y_{i-1} = \phi(x_i) - \phi(x_{i-1}) = \phi'(t_i)\Delta x_i.$$

- Let $M = \sup_{x\in[a,b]} |f(x)|$. Then, if $s_i \in [x_{i-1}, x_i]$, we have

$$\sum_{i=1}^{n} |\phi'(s_i) - \phi'(t_i)|\Delta x_i < \varepsilon.$$

- Hence,

$$\sum_{i=1}^{n} f(s_i)\Delta y_i = \sum_{i=1}^{n} f(s_i)\phi'(t_i)\Delta x_i.$$

$$\left|\sum_{i=1}^{n} f(s_i)\Delta y_i - \sum_{i=1}^{n} f(s_i)\phi'(s_i)\Delta x_i\right| < M\varepsilon.$$

Proof 2/2

- In particular,

$$\sum_{i=1}^{n} f(s_i)\Delta y_i \leq U(P, f\phi') + M\varepsilon.$$

- The last inequality holds for all choices of P (consequently Q), hence

$$U(Q, f) \leq U(P, f\phi') + M\varepsilon \quad \Longrightarrow \quad |U(Q, f) - U(P, f\phi')| \leq M\varepsilon.$$

- Hence

$$\left| \overline{\int_a^b} f(x)\, dx - \overline{\int_a^b} f(x)\phi'(x)\, dx \right| < M\varepsilon.$$

  Since $\varepsilon > 0$ was arbitrary,

$$\overline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\phi'(x)\, dx$$

- The equality for lower integral follows the same way. □

Example

Example

We note that

$$\int_0^2 2x \cdot e^{-x^2}\, dx = \int_0^4 e^{-x}\, dx.$$

Taking $\phi : [0,2] \to [0,4]$ given by $\phi(x) = x^2$, and $f(x) = e^{-x}$, we see that

$$\int_0^2 \underbrace{(2x)}_{\phi'(x)} \cdot \underbrace{e^{-x^2}}_{f(\phi(x))}\, dx = \int_0^4 e^{-x}\, dx.$$

Integration and differentiation

Integration and differentiation

Let $f \in \mathcal{R}([a,b])$ and

$$F(x) = \int_a^x f(y)\,dy \quad \text{for} \quad x \in [a,b],$$

then F is continuous on $[a,b]$. Furthermore, if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

Proof. Let $\varepsilon > 0$. Suppose that $|f(x)| \le M$. Then for all $a \le x_1 \le x_2 \le b$ we have

$$|F(x_1) - F(x_2)| \le \int_{x_1}^{x_2} |f(y)|\,dy \le M(x_2 - x_1).$$

Therefore, $|F(x_1) - F(x_2)| < \varepsilon$ if $|x_1 - x_2| < \frac{\varepsilon}{M}$. Hence F is **continuous**.

Proof

Now suppose f is continuous at x_0. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(x_0) - f(t)| < \varepsilon \quad \text{if} \quad |x_0 - t| < \delta.$$

Hence, if

$$x_0 - \delta \leq s \leq x_0 \leq t \leq x_0 + \delta,$$

then

$$\begin{aligned} \left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| &= \left| \frac{F(t) - F(s)}{t - s} - \frac{1}{t - s} \int_s^t f(x_0)\, dy \right| \\ &= \left| \frac{1}{t - s} \int_s^t (f(y) - f(x_0))\, dy \right| \leq \frac{t - s}{t - s} \varepsilon = \varepsilon. \end{aligned}$$

It follows that $F'(x_0) = f(x_0)$ and we are done. □

The fundamental theorem of calculus

The fundamental theorem of calculus

If $f \in \mathcal{R}([a,b])$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let $\varepsilon > 0$. Choose a partition P such that $U(P,f) - L(P,f) < \varepsilon$. The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$. Hence,

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) = F(b) - F(a).$$

This completes the proof, since

$$\left| F(b) - F(a) - \int_a^b f(y)\,dy \right| < \varepsilon. \qquad \square$$

Example

Example 1

Let

$$h(x) = \int_0^x e^{-y^3} \sin(5y)\, dy.$$

Calculate the derivative of h.

Solution. By the previous theorem we have

$$h'(x) = e^{-x^3} \sin(5x).$$

Example

Example 2

Let

$$h(x) = \int_0^{x^2} \frac{1}{y^2 + e^{y^6}}\, dy.$$

Calculate the derivative of h.

Solution. Let us denote

$$F(x) = \int_0^{x} \frac{1}{y^2 + e^{y^6}}\, dy.$$

By the previous theorem

$$F'(x) = f(x) = \frac{1}{x^2 + e^{x^6}}.$$

Note that $h(x) = F(x^2)$. Therefore,

$$h'(x) = 2xF'(x) = 2xf(x) = \frac{2x}{x^2 + e^{x^6}}.$$

Theorem (integration by parts)

Theorem (integration by parts)

Suppose F and G are differentiable functions on $[a, b]$, and $F' = f \in \mathcal{R}([a, b])$, and $G' = g \in \mathcal{R}([a, b])$. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx.$$

Proof. Let

$$H(x) = F(x)G(x).$$

By the chain rule,

$$H'(x) = F(x)g(x) + f(x)G(x).$$

Finally, we apply the previous theorem to H. □