Lecture 23

Uniform Convergence of a Sequence of Functions, Uniform Convergence and Differentiation, Series of Functions, The Weierstrass Approximation Theorem

MATH 411H FALL 2025

November 20, 2025

Pointwise convergence

Pointwise convergence

For each $n \in \mathbb{N}$ let $f_n : A \to \mathbb{R}$, where $A \subseteq X$. The sequence $(f_n)_{n\in\mathbb{N}}$ of functions **converges pointwise** on A to a function f if, for all $x \in A$, the sequence of real numbers $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$. We write

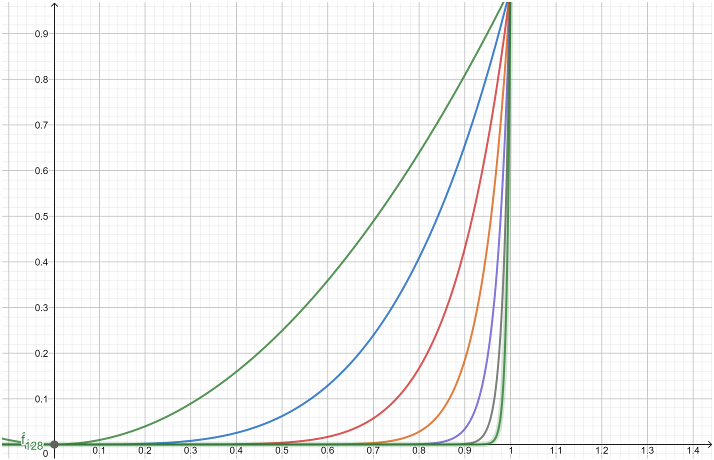
$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{or} \quad f_n \xrightarrow[n\to\infty]{} f.$$

Example 1

Let $g_n(x) = x^n$ for $x \in [0,1]$, then

$$\lim_{n\to\infty} g_n(x) = \begin{cases} 0 \text{ if } x \in [0,1), \\ 1 \text{ if } x = 1. \end{cases}$$

Graphs of $g_2, g_4, g_8, g_{16}, g_{32}, g_{64}, g_{128}$



Examples

Example 2

Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$. Consider $s_N(x) = \sum_{n=0}^{N} f_n(x)$, then

$$s_N(x) \xrightarrow[N\to\infty]{} f(x),$$

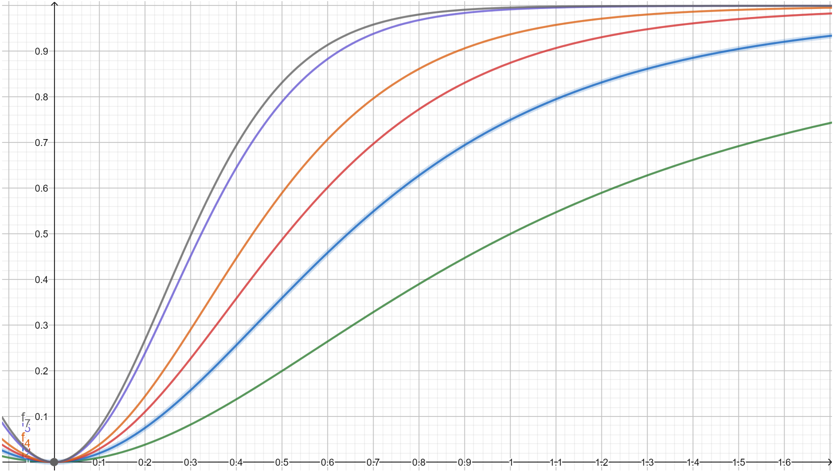
where

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 + x^2 & \text{if } x \neq 0 \end{cases}$$

since if $x \neq 0$ one has

$$\lim_{N\to\infty} s_N(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \frac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2.$$

Graphs of $s_1, s_2, s_3, s_4, s_5, s_6, s_7$



Examples

Example 3

Let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$, then $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Also we see that $f'(x) = 0$, but

$$f_n'(x) = \sqrt{n}\cos(nx)$$

does not converge to $f'(x)$ since

$$f'(0) = \sqrt{n} \xrightarrow[n\to\infty]{} +\infty.$$

Uniform convergence

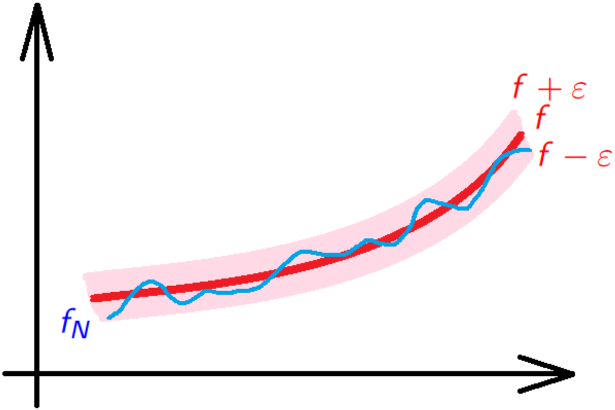
Uniform convergence

We say that a sequence of functions $(f_n)_{n\in\mathbb{N}}$ **converges uniformly** on E to a function f is for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in \mathbb{N}$. We shall write $f_n \underset{n\to\infty}{\rightrightarrows} f$ if $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Remark

Clearly every uniformly convergent sequence is pointwise convergent.

Uniform convergence - picture



Theorem

Theorem

The sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on E converges uniformly on E iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$|f_n(x) - f_m(x)| \leq \varepsilon \quad \text{for all} \quad x \in E.$$

Proof ($\Longrightarrow$). Suppose $(f_n)_{n\in\mathbb{N}}$ converges uniformly on E and let f be the limit function. Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2} \quad \text{for all} \quad x \in E.$$

Thus

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if $n, m \geq N$ and $x \in E$.

Proof

Proof (⟸). Conversely, suppose that Cauchy criterion holds.

- Then $(f_n(x))_{n\in\mathbb{N}}$ converges for every $x \in E$ to a limit, which we will call $f(x)$.
- Thus $f_n \xrightarrow[n\to\infty]{} f$ pointwise.
- We will show that the convergence is uniform.
- Let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ so that $n, m \geq N$ implies

$$|f_m(x) - f_n(x)| \leq \varepsilon \quad \text{for all} \quad n \in E.$$

- Fix n and let $m \to \infty$. Thus

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all $n \geq N$ and $x \in E$, and we are done. □

Uniform convergence of a series

Theorem

$$f_n \underset{n\to\infty}{\rightrightarrows} f \quad \text{on} \quad E \quad \Longleftrightarrow \quad M_n = \sup_{x\in E} |f_n(x) - f(x)| \underset{n\to\infty}{\longrightarrow} 0.$$

Proof. It is an immediate consequence of the definition.

Uniform convergence of a series

We say that the series

$$\sum_{n=0}^{\infty} f_n(x)$$

converges uniformly on E if the sequence

$$s_n(x) = \sum_{k=0}^{n} f_k(x) \quad \text{converges uniformly on } E.$$

Theorem

Theorem

Suppose that $f_n : E \to \mathbb{R}$ and $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and $x \in E$. Then $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on E if

$$\sum_{n=0}^{\infty} M_n < \infty.$$

Proof. Let $\varepsilon > 0$ and $\sum_{k=n+1}^{m} M_k \leq \varepsilon$ if $m, n \geq N$ for some $N \in \mathbb{N}$. Then

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} M_k \leq \varepsilon$$

for all $x \in E$ and $m, n \geq N$. □

Interchange limit theorem

Theorem

Suppose that $f_n \underset{n\to\infty}{\rightrightarrows} f$ on E. Let x be a limit point of E and suppose that $\lim_{t\to x} f_n(t) = A_n$. Then $(A_n)_{n\in\mathbb{N}}$ converges and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n.$$

In other words, we may write

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

Proof. Let $\varepsilon > 0$ be given. Since $f_n \underset{n\to\infty}{\rightrightarrows} f$ there is $N \in \mathbb{N}$ such that $m, n \geq N$ implies

$$|f_m(t) - f_n(t)| \leq \varepsilon \quad \text{for all} \quad t \in E. \qquad (*)$$

Proof

- Letting $t \to x$ in (*) we see for all $n, m \geq N$ that
$$|A_n - A_m| \leq \varepsilon.$$
- Thus $(A_n)_{n\in\mathbb{N}}$ is Cauchy. Hence $A_n \xrightarrow[n\to\infty]{} A$ for some $A \in \mathbb{R}$. Next
$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$
- We first choose $n \in \mathbb{N}$ so that $|f(t) - f_n(t)| \leq \frac{\varepsilon}{3}$ for all $t \in E$, and $|A_n - A| \leq \frac{\varepsilon}{3}$.
- For this $n \in \mathbb{N}$, we choose an open set V containing x such that
$$|f_n(t) - A_n| \leq \frac{\varepsilon}{3}$$
if $t \in V \cap E$ and $t \neq x$. Hence
$$|f(t) - A| \leq \varepsilon$$
provided that $t \in V \cap E$ and $t \neq x$. □

Important theorems

Theorem

If $f_n : E \to \mathbb{R}$ is continuous and $f_n \underset{n\to\infty}{\rightrightarrows} f$ on E then f is continuous on E.

Proof. It follows from the previous theorem. □

Remark

The converse in the theorem above is not true.

Theorem

Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions differentiable on $[a, b]$ and such that $(f_n(x_0))_{n\in\mathbb{N}}$ converges for some point $x_0 \in [a, b]$. If $(f_n')_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$ then $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a, b]$ to a function f and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad \text{for} \quad x \in [a, b].$$

Proof 1/2

Proof. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ so that $n, m \geq N$ implies

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \quad \text{and} \quad |f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)} \quad \text{for} \quad t \in [a, b].$$

- By the mean-value theorem applied to $f_n - f_m$ we have

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x - t|\varepsilon}{2(b-a)} \leq \frac{\varepsilon}{2} \qquad (*)$$

  for any $x, t \in [a, b]$ if $m, n \geq N$.
- The inequality

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

  implies that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$ and $x \in [a, b]$, so $(f_n)_{n \in \mathbb{N}}$ converges uniformly on $[a, b]$.
- Let

$$f(x) = \lim_{n \to \infty} f_n(x), \quad a \leq x \leq b.$$

Proof 2/2

- Fix a point $x \in [a, b]$ and define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad \phi(t) = \frac{f(t) - f(x)}{t - x}, \quad t \in [a, b], \quad t \neq x$$

- Then $\lim_{t \to x} \phi_n(t) = f_n'(x)$ for all $n \in \mathbb{N}$. Inequality (*) also shows

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b-a)} \quad \text{if} \quad n, m \geq N.$$

- Thus $(\phi_n)_{n \in \mathbb{N}}$ converges uniformly for $x \neq t$. Since $f_n \underset{n \to \infty}{\rightrightarrows} f$ thus

$$\lim_{n \to \infty} \phi_n(t) = \phi(t) \quad \text{for} \quad a \leq x \leq b, \quad t \neq x.$$

- By the previous theorem

$$\lim_{n \to \infty} f_n'(x) = \lim_{n \to \infty} \lim_{t \to x} \phi_n(t) = \lim_{t \to x} \lim_{n \to \infty} \phi_n(t) = \lim_{t \to x} \phi(t) = f'(x). \quad \square$$

Continuous nowhere differentiable function

Theorem

There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere differentiable.

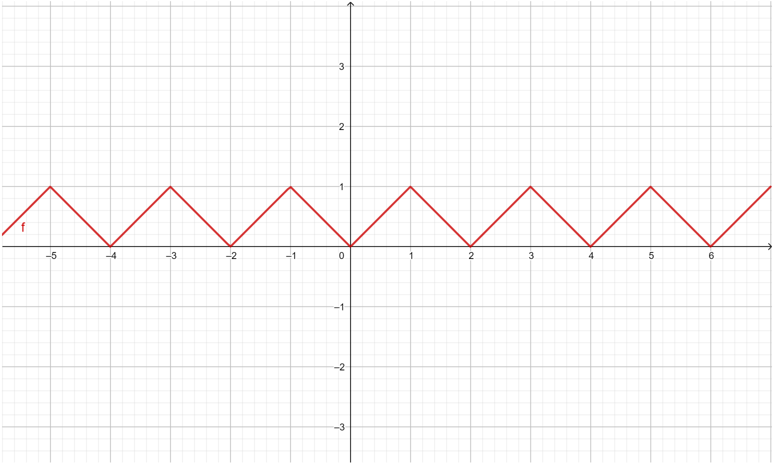
Proof. Let $\phi(x) = |x|$ on $[-1, 1]$ and extend the definition of $\phi(x)$ to all $x \in \mathbb{R}$ by setting

$$\phi(x) = \phi(x+2)$$

for all $x \in \mathbb{R}$. Then

$$|\phi(s) - \phi(t)| \leq |s - t| \quad \text{for all} \quad s, t \in \mathbb{R}.$$

Graph of ϕ



Proof 1/2

- Define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x).$$

- Since $0 \leq \phi(x) \leq 1$ then the series converges uniformly on $\mathbb{R}$ and f is continuous.
- Now fix $x \in \mathbb{R}$ and $m \in \mathbb{N}$ and put

$$\delta_m = \pm \frac{1}{2} 4^{-m},$$

where the sign is chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$. This can be done since $4^m |\delta_m| = \frac{1}{2}$.
- Define

$$\gamma_n = \frac{\phi(4^n(x + \delta_m)) - \phi(4^n x)}{\delta_m}.$$

Proof 2/2

- When $n > m$ then $4^n\delta_m$ is an integer so that $\gamma_n = 0$.
- When $0 \leq n \leq m$, then $|\gamma_n| \leq 4^n$. Since $|\gamma_m| = 4^m$ we conclude

$$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| = \left|\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n\right|$$
$$\geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m - 1) \xrightarrow[m\to\infty]{} \infty$$

and $\delta_m \xrightarrow[m\to\infty]{} 0$ thus $f'(x)$ does not exists. □

Weierstrass theorem

Weierstrass theorem

Let $-\infty < a < b < \infty$. Every continuous $f : [a, b] \to \mathbb{R}$ can be uniformly approximated by polynomials. In other words, for every continuous $f : [a, b] \to \mathbb{R}$ there is a sequence of polynomials $(p_n(f))_{n\in\mathbb{N}}$ so that

$$\sup_{x\in[a,b]} |p_n(f)(x) - f(x)| \xrightarrow[n\to\infty]{} 0.$$

Proof. Using a linear transformation

$$[a, b] \ni t \to \frac{s-a}{s-b}$$

we can assume that $[a, b] = [0, 1]$. Fix a continuous $f : [0, 1] \to \mathbb{R}$, and set

$$p_n(f)(t) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) t^k (1-t)^{n-k} \quad \text{for} \quad t \in [0, 1].$$

Proof 1/3

- We show that $p_n(f) \underset{n\to\infty}{\rightrightarrows} f$. Let $\varepsilon > 0$ be given. Since f is uniformly continuous on $[0,1]$ so there is $\delta > 0$ so that

$$|f(t) - f(s)| < \varepsilon \text{ if } |s - t| < \delta.$$

- Note that

$$\sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} = 1.$$

- Hence

$$|f(t) - p_n(f)(t)| \leq \sum_{k=0}^{n} \binom{n}{k} \left| f(t) - f\left(\frac{k}{n}\right) \right| t^k (1-t)^{n-k}.$$

Proof 2/3

- Let $M = \sup_{x\in[0,1]} |f(x)|$, and note that

$$
\begin{aligned}
|f(t) - p_n(f)(t)| &\le \varepsilon \sum_{\substack{k=0 \\ |t-k/n|<\delta}}^{n} \binom{n}{k} t^k (1-t)^{n-k} \\
&\quad + 2M \sum_{\substack{k=0 \\ |t-k/n|\ge\delta}}^{n} \binom{n}{k} t^k (1-t)^{n-k} \\
&\le \varepsilon + 2M\delta^{-2} \sum_{k=0}^{n} \binom{n}{k} (t-k/n)^2 t^k (1-t)^{n-k}.
\end{aligned}
$$

Proof 3/3

- So we have to estimate

$$2M\delta^{-2}\sum_{k=0}^{n}\binom{n}{k}(t-k/n)^2 t^k(1-t)^{n-k}.$$

- Then, using the identity

$$\sum_{k=0}^{n}\binom{n}{k}(t-k/n)^2 t^k(1-t)^{n-k}=\frac{t(1-t)}{n}$$

we obtain

$$2M\delta^{-2}\sum_{k=0}^{n}\binom{n}{k}(t-k/n)^2 t^k(1-t)^{n-k}\leq\frac{2M\delta^{-2}}{n}$$

and we are done. □

Analytic functions

Analytic functions

Functions which can be represented as power series

$$\sum_{n=0}^{\infty} c_n x^n, \quad x \in \mathbb{R}, \qquad (*)$$

or more generally

$$\sum_{n=0}^{\infty} c_n (x-a)^n, \quad x, a \in \mathbb{R}, \qquad (**)$$

are called **analytic functions.**

Remark

- If (**) converges for $|x-a| < R$ for some $R \in (0, \infty]$, f is said to be expanded in a power series about the point $x = a$.
- As a matter of convenience, we shall often take $a = 0$ without any loss of generality and work with (*).

Differentiability of power series

Theorem

Suppose that the series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

converges for $|x| < R$*, then it converges uniformly on* $[-R+\varepsilon, R-\varepsilon]$*, no matter which* $\varepsilon > 0$ *is chosen. Moreover, the function* f *is continuous and differentiable in* $(-R, R)$*, and*

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad \text{for} \quad |x| < R.$$

Proof. Let $\varepsilon > 0$ be given. For $|x| \leq R - \varepsilon$, by the root test, we have

$$\sum_{n=0}^{\infty} |c_n x^n| \leq \sum_{n=0}^{\infty} |c_n (R-\varepsilon)^n| < \infty,$$

Proof

- Thus the sequence
$$f_N(x) = \sum_{n=0}^{N} c_n x^n$$
converges absolutely to $f(x)$ on $[-R+\varepsilon, R-\varepsilon]$.
- Note that $(f_N)_{N\in\mathbb{N}}$ is a sequence of differentiable functions on $(-R, R)$ that converges to $f(x)$ for any $x \in (-R, R)$.
- Moreover, $(f'_N)_{N\in\mathbb{N}}$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$, since $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=1}^{\infty} n c_n x^{n-1}$ have the same intervals of convergence as
$$\limsup_{n\to\infty} \sqrt[n]{n|c_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$
- Thus
$$f'(x) = \lim_{N\to\infty} f'_N(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}$$
and clearly f is continuous as a differentiable function. □

Power series are differentiable infinitaley many times

Corollary

Under the assumption of the previous theorem

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \textit{for} \quad |x| < R$$

has derivatives of all orders in $(-R, R)$, *which are given by*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdot \ldots \cdot (n-k+1) c_n x^{n-k}, \quad \textit{for} \quad |x| < R.$$

In particular, we have

$$f^{(k)}(0) = k! c_k.$$

Here $f^{(0)} = f$ *and* $f^{(k)}$ *is the* k*-th derivative of* f *for* $k \in \mathbb{N}$.

Convergence at the endpoint

Theorem

Suppose that the series

$$s = \sum_{n=0}^{\infty} c_n$$

converges, and set

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{for} \quad |x| < 1.$$

Then

$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n.$$

Proof. We will use the Abel summation formula. Let $s_{-1} = 0$ and

$$s_n = \sum_{k=0}^{n} c_k \quad \text{for} \quad n \in \mathbb{N} \cup \{0\}, \qquad \text{and} \qquad s = \lim_{n \to \infty} s_n.$$

Proof

- Note that

$$\sum_{n=0}^{m} c_n x^n = \sum_{n=0}^{m} (s_n - s_{n-1}) x^n = (1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m.$$

- For $|x| < 1$ if we take $m \to \infty$ we obtain

$$f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n.$$

- Given $\varepsilon > 0$ we choose $N \in \mathbb{N}$ such that $n > N$ implies $|s_n - s| < \frac{\varepsilon}{2}$. Since $(1-x) \sum_{n=0}^{\infty} x^n = 1$ we may write

$$|f(x) - 1| = \left| (1-x) \sum_{n=0}^{\infty} (s_n - s) x^n \right| \leq (1-x) \sum_{n=0}^{N} |s_n - s||x|^n + \frac{\varepsilon}{2}.$$

- If $x > 1 - \delta$ for a suitably chosen $\delta > 0$ we have

$$(1-x) \sum_{n=0}^{N} |s_n - s||x|^n < \frac{\varepsilon}{2}. \qquad \square$$

Product of converging series

Remark

If $\sum_{n=0}^{\infty} a_n$ converges absolutely, $\sum_{n=0}^{\infty} a_n = A$, and $\sum_{n=0}^{\infty} b_n = B$, and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad \text{for} \quad n = 0, 1, 2, \ldots.$$

Then $\sum_{k=0}^{\infty} c_k = AB$.

By the previous theorem this result can be extended as follows:

Theorem

If the series $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$, and $\sum_{n=0}^{\infty} c_n = C$, converge and

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \quad \textit{for} \quad n = 0, 1, 2, \ldots.$$

Then $AB = C$.

Proof

Proof. For $0 \le x \le 1$ we let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n x^n, \qquad h(x) = \sum_{n=0}^{\infty} c_n x^n.$$

- If $0 \le x < 1$ these series converge absolutely, thus by the previous remark we obtain

$$f(x) \cdot g(x) = h(x), \quad \text{for} \quad 0 \le x < 1.$$

- By the previous theorem we may conclude that

$$\lim_{x \to 1} f(x) = A, \qquad \lim_{x \to 1} g(x) = B, \qquad \lim_{x \to 1} h(x) = C.$$

- Hence we obtain $AB = C$. □

Uniqueness of the power series expansions

Theorem

Suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge in the segment $S = (-R, R)$. Let

$$E = \left\{ x \in S : \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n \right\}.$$

If E as a limit point in S, then $a_n = b_n$ for all $n \in \mathbb{N} \cup \{0\}$, and $E = S$.

Proof. Put $c_n = a_n - b_n$ and let

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad \text{for} \quad x \in S.$$

Then $f(x) = 0$ on E. We prove that $f(x) = 0$ on S.

Proof

- Let A be the set of all limit points of E in S, and let B consist of all other points of S. It is clear from the definition of "limit point" that B is open. Suppose we can prove that A is open.
- Then A and B are disjoint open sets. Hence they are separated. Since $S = A \cup B$, and S is connected, one of A and B must be empty. By hypothesis, A is not empty. Hence B is empty, and $A = S$. Since f is continuous in S, $A \subseteq E$.
- Thus $E = S$, and $c_k = \frac{f^{(k)}(0)}{k!} = 0$ for $k \in \mathbb{N} \cup \{0\}$ which is the desired conclusion.
- Now we have to prove that A is open. If $x_0 \in A$, then it is easy to show that

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n, \qquad \text{for} \qquad |x - x_0| < R - |x_0|.$$

Proof

- We claim that $d_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$. Otherwise, let $k \in \mathbb{N} \cup \{0\}$ be the smallest integer such that $d_k \neq 0$. Then

$$f(x) = (x - x_0)^k g(x), \quad \text{for} \quad |x - x_0| < R - |x_0|, \qquad (*)$$

where

$$g(x) = \sum_{m=0}^{\infty} d_{n+m}(x - x_0)^m.$$

- Since g is continuous at x_0 and $g(x_0) = d_k \neq 0$, there exists a $\delta > 0$ such that $g(x) \neq 0$ if $|x - x_0| < \delta$.
- It follows from (*) that $f(x) \neq 0$ for $0 < |x - x_0| < \delta$. But this contradicts the fact that $x_0 \in A$ is a limit point of E, which ensures by continuity of f that $f(x_0) = 0$.
- Thus we have proved that $d_n = 0$ for all $n \in \mathbb{N} \cup \{0\}$, so $f(x) = 0$ on a neighborhood of $x_0 \in A$. This show that A is open as desired. □