Lecture 24

Applications of calculus: Fundamental theorem of algebra, Stirling's formula, Equidistribution theorem of Weyl, Transcendence of the Euler's number

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Fibonacci sequence

Fibonacci sequence

The Fibonacci sequence $(f_n)_{n\in\mathbb{N}}$ is defined by

$$f_0 = 0, \quad f_1 = 1,$$

$$f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2.$$

Example

$$f_2 = 0 + 1 = 1,$$

$$f_3 = 1 + 1 = 2,$$

$$f_4 = 1 + 2 = 3,$$

$$f_5 = 2 + 3 = 5,$$

$$f_6 = 8, \quad f_7 = 13, \quad f_8 = 21.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 1/4

- Consider

$$\begin{aligned}\sum_{n=0}^{\infty} f_n x^n &= x + \sum_{n=2}^{\infty}(f_{n-1}+f_{n-2})x^n \\ &= x + x\sum_{n=2}^{\infty} f_{n-1}x^{n-1} + x^2\sum_{n=2}^{\infty} f_{n-2}x^{n-2} \\ &= (x+x^2)\sum_{n=0}^{\infty} f_n x^n + x.\end{aligned}$$

- Denoting $F(x)=\sum_{n=0}^{\infty} f_n x^n$ we have

$$F(x) = x + F(x)(x+x^2),$$

so

$$F(x) = \frac{x}{1-x-x^2}.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 2/4

- Then
$$1 - x - x^2 = -(x+\phi)(x+\psi),$$
where
$$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}.$$

- Then
$$F(x) = -\frac{x}{(x+\phi)(x+\psi)} = \frac{A}{x+\phi} + \frac{B}{x+\psi},$$
which is equivalent to
$$-x = A(x+\psi) + B(x+\phi).$$

- Hence
$$A = \frac{-\phi}{\sqrt{5}} = \frac{1+\sqrt{5}}{2\sqrt{5}}, \quad B = \frac{\psi}{\sqrt{5}} = \frac{1-\sqrt{5}}{2\sqrt{5}}.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 3/4

- So

$$F(x) = \frac{1}{\sqrt{5}}\left(\frac{\psi}{x+\psi} - \frac{\phi}{x+\phi}\right).$$

- Recall that for $|x| < 1$ we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

- Therefore

$$\frac{\psi}{x+\psi} = \frac{1}{1+\frac{x}{\psi}} = \frac{1}{1-x\phi} = \sum_{n=0}^{\infty} \phi^n x^n,$$

$$\frac{\phi}{x+\phi} = \sum_{n=0}^{\infty} \psi^n x^n.$$

Formula for $(f_n)_{n\in\mathbb{N}}$ - discussion 4/4

- Finally, we have

$$\sum_{n=0}^{\infty} f_n x^n = F(x)$$

$$= \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{\psi}{x+\psi} - \frac{\phi}{x+\phi}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\phi^n x^n - \sum_{n=0}^{\infty}\psi^n x^2\right) = \sum_{n=0}^{\infty}\frac{1}{\sqrt{5}}(\phi^n - \psi^n)x^n.$$

- Thus the formula for $(f_n)_{n\in\mathbb{N}}$ is given by

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra)

Suppose $a_0, a_1, \ldots, a_n \in \mathbb{C}$ *and* $a_n \neq 0$ *with* $n \in \mathbb{N}$. *Let*

$$P(z) = \sum_{k=0}^{n} a_k z^k$$

then $P(z) = 0$ *for some complex number* $z \in \mathbb{C}$.

Proof. Without loss of generality, we assume that $a_n = 1$ and let

$$\mu = \inf_{z \in \mathbb{C}} |P(z)|.$$

If $|z| = R$, then by the triangle inequality we have

$$\begin{aligned} |P(z)| &\geq \left| |a_n z^n| - |a_{n-1} z^{n-1}| - \ldots - |a_0| \right| \\ &= R^n \cdot (1 - |a_{n-1}| R^{-1} - \ldots - |a_0| R^{-n}). \end{aligned}$$

The right-hand side tends to ∞ as $R \to \infty$.

Proof

- Hence, by the definition of divergence, there is $R_0 > 0$ such that

$$|P(z)| > \mu \quad \text{if} \quad |z| > R_0.$$

- Since $|P|$ is continuous on the closed circle with center 0 and radius R_0, it attains its minimum value at some $z_0 \in \mathbb{C}$ so that $|z_0| \leq R_0$.
- We claim that $\mu = 0$. If not, set $Q(z) = \frac{P(z+z_0)}{P(z_0)}$ then Q is a nonconstant polynomial with $Q(0) = 1$ and

$$|Q(z)| \geq 1 \quad \text{for all} \quad z \in \mathbb{C}.$$

- There is a smallest integer $1 \leq k \leq n$ such that

$$Q(z) = 1 + b_k z^k + \ldots + b_n z^n, \qquad b_k \neq 0.$$

- Then there is $\theta \in \mathbb{R}$ such that

$$e^{ik\theta} b_k = -|b_k|.$$

Proof

- If $r > 0$ and $r^k|b_k| < 1$, then the equation above implies that

$$|1 + b_k r^k e^{ik\theta}| = |1 - r^k|b_k|| = 1 - r^k|b_k|,$$

so that

$$|Q(re^{i\theta})| \leq 1 - r^k\{|b_k| - r|b_{k+1}| - \ldots - r^{n-1}|b_n|\}.$$

- For sufficiently small r the expression above in braces is positive. Hence

$$|Q(re^{i\theta})| < 1.$$

Thus $\mu = 0$ and $P(z_0) = 0$. □

Stirling's formula

Theorem

For $n \in \mathbb{N}$, we have

$$n! = \sqrt{2\pi} n^{n+1/2} e^{-n} e^{r_n},$$

where r_n satisfies the double inequality

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}.$$

The usual textbook proofs replace the first inequality above by the weaker inequality

$$r_n > 0,$$

or

$$r_n > \frac{1}{12n+6}.$$

Proof

Proof. Let

$$S_n = \log(n!) = \sum_{p=1}^{n-1} \log(p+1)$$

and write

$$\log(p+1) = A_p + b_p - \varepsilon_p,$$

where

$$A_p = \int_p^{p+1} \log x \, dx,$$
$$b_p = [\log(p+1) - \log p]/2,$$
$$\varepsilon_p = \int_p^{p+1} \log x \, dx - [\log(p+1) + \log p]/2.$$

Proof

The partition of $\log(p+1)$, regarded as the area of a rectangle with base $(p, p+1)$ and height $\log(p+1)$, into a curvilinear area, a triangle, and a small sliver is suggested by the geometry of the curve $y = \log x$. Then

$$S_n = \sum_{p=1}^{n-1} (A_p + b_p - \varepsilon_p) = \int_1^n \log x \, dx + \frac{1}{2} \log n - \sum_{p=1}^{n-1} \varepsilon_p.$$

Since $\int \log x \, dx = x \log x - x$ we can write

$$S_n = (n + 1/2) \log n - n + 1 - \sum_{p=1}^{n-1} \varepsilon_p,$$

where

$$\varepsilon_p = \frac{2p+1}{2} \log \left(\frac{p+1}{p} \right) - 1.$$

Proof

Using the well known series expansions

$$\log\left(\frac{1+x}{1-x}\right) = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

valid for $|x| < 1$, and setting $x = (2p+1)^{-1}$, so that $(1+x)/(1-x) = (p+1)/p$, we find that

$$\varepsilon_p = \sum_{k=0}^{\infty} \frac{1}{(2k+3)(2p+1)^{2k+2}}.$$

We can therefore bound ε_p above:

$$\varepsilon_p < \frac{1}{3(2p+1)^2} \sum_{k=0}^{\infty} \frac{1}{(2p+1)^{2k}} = \frac{1}{12}\left(\frac{1}{p} - \frac{1}{p+1}\right),$$

Proof

Similarly, we bound ε_p below:

$$\varepsilon_p > \frac{1}{3(2p+1)^2} \sum_{k=0}^{\infty} \frac{1}{[3(2p+1)^2]^k} = \frac{1}{3(2p+1)^2} \frac{1}{1 - \frac{1}{3(2p+1)^2}}$$
$$> \frac{1}{12}\left(\frac{1}{p+1/12} - \frac{1}{p+1+1/12}\right).$$

Now define

$$B = \sum_{p=1}^{\infty} \varepsilon_p, \qquad r_n = \sum_{p=n}^{\infty} \varepsilon_p,$$

where from the lower and upper bound for ε_p we have

$$1/13 < B < 1/12.$$

Then we can write

$$S_n = (n+1/2)\log n - n + 1 - \sum_{p=1}^{n-1} \varepsilon_p$$
$$= (n+1/2)\log n - n + 1 - B + r_n,$$

or, setting $C = e^{1-B}$, as

$$n! = Cn^{n+1/2}e^{-n}e^{r_n},$$

where r_n satisfies

$$1/(12n+1) < r_n < 1/(12n).$$

The constant C, lies between $e^{11/12}$ and $e^{12/13}$, may be shown to have the value $\sqrt{2\pi}$. This completes the proof. □

Equidistributed sequences

Definition

A sequence $(a_k)_{k\in\mathbb{N}\cup\{0\}} \subseteq [0,1]$ is called equidistributed if for any real numbers a, b with $0 \le a < b \le 1$ one has

$$\lim_{N\to\infty} \frac{\#\{k \in [0,N)\cap\mathbb{Z} \colon a_k \in [a,b]\}}{N} = b - a.$$

Weyl's equidistribution theorem

Theorem

The following statements are equivalent:

(a) *The sequence* $(a_k)_{k\in\mathbb{N}\cup\{0\}} \subseteq [0,1]$ *is equidistributed.*

(b) *For every* $m \in \mathbb{Z}\setminus\{0\}$ *we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i m a_k} = 0.$$

(c) *For every continuous function* $f \in C([0,1],\mathbb{C})$ *we have that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} f(a_k) = \int_0^1 f(x)dx.$$

Proof

We first prove the equivalence of (a) and (c). Assume that (c) holds, and fix $0 \le a < b \le 1$. Given a sufficiently small $\varepsilon > 0$, we define continuous functions $f^-, f^+ : [0,1] \to [0,1]$ that approximate the indicator function $\mathbb{1}_{[a,b]}$ by

$$f^+(x) = \begin{cases} 1 & \text{if } a \le x \le b; \\ \varepsilon^{-1}(x-(a-\varepsilon)) & \text{if } \max\{0, a-\varepsilon\} \le x < a; \\ \varepsilon^{-1}((b+\varepsilon)-x) & \text{if } b < x \le \max\{b+\varepsilon, 1\}; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f^-(x) = \begin{cases} 1 & \text{if } a+\varepsilon \le x \le b-\varepsilon; \\ \varepsilon^{-1}(x-a) & \text{if } a \le x < a+\varepsilon; \\ \varepsilon^{-1}(b-x) & \text{if } b-\varepsilon < x \le b; \\ 0 & \text{otherwise.} \end{cases}$$

Proof

Notice that $f^-(x) \leq \mathbb{1}_{[a,b]}(x) \leq f^+(x)$ for all $x \in [0,1]$, and

$$\int_0^1 (f^+(x) - f^-(x))dx \leq 2\varepsilon.$$

It follows that

$$\frac{1}{N}\sum_{k=0}^{N-1} f^-(a_k) \leq \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) \leq \frac{1}{N}\sum_{k=0}^{N-1} f^+(a_k).$$

By (c) we have

$$\begin{aligned} b - a - 2\varepsilon \leq \int_0^1 f^-(x)dx &\leq \liminf_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) \\ &\leq \limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) \leq \int_0^1 f^+(x)dx \leq b - a + 2\varepsilon. \end{aligned}$$

Proof

Thus (a) is proved, since

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) = \limsup_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} \mathbb{1}_{[a,b]}(a_k) = b-a.$$

Assume that (a) holds. Given a continuous function f on $[0,1]$ and given $\varepsilon > 0$, we find a step function, i.e.

$$g = \sum_{j=1}^{m} c_j \mathbb{1}_{I_j},$$

such that $\|f-g\|_\infty < \varepsilon/3$, where $c_j \in \mathbb{C}$ and $I_j \subseteq [0,1]$ are intervals. Since g is a finite linear combination of indicator functions, there is an $N_0 \in \mathbb{N}$ such that for $N \geq N_0$ we have

$$\Big|\frac{1}{N}\sum_{k=0}^{N-1} g(a_k) - \int_0^1 g(x)dx\Big| < \varepsilon/3.$$

Proof

Since

$$\Big|\int_0^1 f(x)dx - \int_0^1 g(x)dx\Big| \le \|f-g\|_\infty < \varepsilon/3$$

and

$$\Big|\frac{1}{N}\sum_{k=0}^{N-1} g(a_k) - \frac{1}{N}\sum_{k=0}^{N-1} f(a_k)\Big| \le \|f-g\|_\infty < \varepsilon/3,$$

it follows that for $N \ge N_0$ we have

$$\Big|\frac{1}{N}\sum_{k=0}^{N-1} f(a_k) - \int_0^1 f(x)dx\Big| < \varepsilon,$$

thus (c) holds.

Proof

We now prove the equivalence of (b) and (c). In one direction this is clear. To see that (b) implies (c) we fix a continuous function f on $[0,1]$. Then for a given $\varepsilon > 0$ by Weierstrass theorem we pick a trigonometric polynomial p such that $\|f - p\|_\infty < \varepsilon/3$. Since

$$p(x) = \sum_{m=-M}^{M} c_m e^{2\pi i m x}$$

for some $M \in \mathbb{N}$ and $c_m \in \mathbb{C}$, then by (b) we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} p(a_k) = c_0 = \int_0^1 p(x)dx.$$

Hence a 3-epsilon argument completes the proof, as we have

$$\left|\frac{1}{N} \sum_{k=0}^{N-1} f(a_k) - \int_0^1 f(x)dx\right| < \varepsilon. \qquad \square$$

Example

Example

- The sequence $(\{k\sqrt{2}\})_{k\in\mathbb{N}\cup\{0\}}$ is equidistributed on $[0,1]$.
- We check this by verifying condition (b) of the previous theorem. Indeed if $m \in \mathbb{Z}\setminus\{0\}$ then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi i m(k\sqrt{2}-\lfloor k\sqrt{2}\rfloor)} = \lim_{N\to\infty}\frac{1}{N}\frac{e^{2\pi i N m\sqrt{2}}-1}{e^{2\pi i m\sqrt{2}}-1} = 0,$$

  since $m\sqrt{2}\in\mathbb{R}\setminus\mathbb{Q}$ thus the denominator never vanishes.
- Naturally, the same conclusion is valid for any other irrational number $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ in place of $\sqrt{2}$.

Gelfand's problem

We will consider the sequence of the first digits of powers of 2. Namely, for $m \in \mathbb{N}$ let

$$d_m = \text{first digit of } 2^m.$$

For instance we have $d_1 = 2$, $d_2 = 4$, $d_3 = 8$, $d_4 = 1$, $d_5 = 3, \ldots$. Here is a list of the first 20 powers of 2:

$$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192,$$
$$16384, 32768, 65536, 131072, 262144, 524288, 1048576.$$

The sequence of the first digits of the first 40 powers of 2 is:

$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1,$$
$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1,$$
$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1,$$
$$2, 4, 8, 1, 3, 6, 1, 2, 5, 1.$$

Gelfand's problem

Gelfand's problem

- Do we ever see 7 or 9? Gelfand's question asks: how often do we see a power of 2 that starts with 7, and with what frequency?
- Surprisingly, we will show here that there are infinitely many $m \in \mathbb{N}$ such that 2^m starts with 7 and we even find their frequency.
- The existence of this frequency will follow from the uniform distribution of multiples of an irrational number modulo 1.

- The crucial observation is that the first digit of 2^m is equal to k if and only if there is a nonnegative integer s such that

$$k10^s \leq 2^m < (k+1)10^s.$$

Gelfand's problem

- Taking logarithms with base 10 we obtain

$$s + \log_{10} k \leq m \log_{10} 2 < s + \log_{10}(k+1),$$

but since $0 \leq \log_{10} k$ and $\log_{10}(k+1) \leq 1$, taking fractional parts we obtain that

$$s = \lfloor m \log_{10} 2 \rfloor$$

and that

$$\log_{10} k \leq m \log_{10} 2 - \lfloor m \log_{10} 2 \rfloor < \log_{10}(k+1). \qquad (*)$$

- Since the number $\log_{10} 2$ is irrational, it follows that the sequence $(\{m \log_{10} 2\})_{m \in \mathbb{N}}$ is dense in $[0, 1]$.
- Therefore, there are infinitely many $m \in \mathbb{N}$ such that (*) holds.

Gelfand's problem

- We recall that for $m \in \mathbb{N}$ we consider

$$d_m = \text{first digit of } 2^m.$$

- Fix an integer $1 \leq k \leq 9$. We will find the frequency in which k appears as a first digit of 2^m, precisely, we would like to find

$$\lim_{N\to\infty} \frac{\#\{m \in \{1,\ldots,N\} : d_m = k\}}{N}.$$

- As we mentioned above it is essential that the first digit of 2^m is equal to k if and only if there is a nonnegative integer s such that

$$k10^s \leq 2^m < (k+1)10^s.$$

- Taking logarithms with base 10 we obtain

$$s + \log_{10} k \leq m \log_{10} 2 < s + \log_{10}(k+1).$$

Gelfand's problem

- Since $0 \leq \log_{10} k$ and $\log_{10}(k+1) \leq 1$, taking fractional parts we obtain that

$$s = \lfloor m \log_{10} 2 \rfloor$$

and that

$$\log_{10} k \leq m \log_{10} 2 - \lfloor m \log_{10} 2 \rfloor < \log_{10}(k+1).$$

- Since the number $\log_{10} 2$ is irrational, the sequence

$$\left(\{m \log_{10} 2\}\right)_{m \in \mathbb{N}}$$

is equidistributed in $[0, 1]$. Using (c) from the previous theorem with $[a, b] = [\log_{10} k, \log_{10}(k+1)]$ we obtain that

$$\lim_{N \to \infty} \frac{\#\{m \in \{1, \dots, N\} : d_m = k\}}{N} = \log_{10}(k+1) - \log_{10} k$$
$$= \log_{10}(1 + 1/k).$$

Gelfand's problem

- This gives the frequency in which k appears as first digit of 2^m. Notice that

$$\sum_{k=1}^{9} \log_{10}(1 + 1/k) = 1,$$

as expected.

- Moreover, the digit with the highest frequency that appears as the first digit in the decimal expansion of the sequence $(2^m)_{m\in\mathbb{N}}$ is 1, while the one with the lowest frequency is 9.

Transcendence of the Euler's number

Definition

A real number is called **algebraic** if it is a root of a polynomial with integer coefficients. Otherwise a real number is called **transcendental.**

Remark

Squaring the circle is a problem proposed by ancient geometers. It was the challenge of constructing a square with the same area as a given circle by using only a finite number of steps with compass and straightedge. In 1882, the task was proven to be impossible, as a consequence of the Lindemann–Weierstrass theorem which proves that π is a transcendental, rather than an algebraic irrational number. It had been known for some decades before then that the construction would be impossible if π were transcendental, but π was not proven transcendental until 1882. A bit simpler is to show that e is transcendental.

Hermite's theorem

Theorem (Hermite's theorem)

The number

$$e = \sum_{k \geq 0} \frac{1}{k!}$$

is transcendental.

Proof. If e were an algebraic number, then we could find a polynomial P with rational coefficients such that

$$P(x) = a_n x^n + \ldots + a_1 x + a_0$$

satisfying $P(e) = 0$. For every prime number $p \in \mathbb{P}$ satisfying $p > n$ and $p > |a_0|$ we define an auxiliary polynomial by setting

$$f_p(x) = \frac{x^{p-1}}{(p-1)!} \prod_{k=1}^{n} (k - x)^p.$$

Proof

We also set

$$F_p(x) = f_p(x) + \sum_{j=1}^{M} f_p^{(j)}(x),$$

where $M = (n+1)p - 1$ is the degree of the polynomial f_p. Since $f_p^{(M+1)}(x) = 0$ we obtain

$$F_p(x) - F_p'(x) = f_p(x),$$

and consequently

$$\left(e^{-x} F_p(x)\right)' = -e^{-x} F_p(x) + e^{-x} F_p'(x) = -e^{-x} f_p(x).$$

By the mean-value theorem we get

$$e^{-x} F_p(x) - F_p(0) = -x e^{-\theta_x x} f_p(\theta_x x)$$

for some $\theta_x \in [0,1]$. Thus

$$F_p(x) - e^x F_p(0) = -x e^{(1-\theta_x)x} f_p(\theta_x x).$$

Proof

If x is fixed and $p \to \infty$, then

$$\lim_{p\to\infty} \left(F_p(x) - e^x F_p(0)\right) = 0,$$

since for every $y \in \mathbb{R}$ we have $\lim_{n\to\infty} \frac{y^n}{n!} = 0$. We also obtain

$$\lim_{p\to\infty} \sum_{k=0}^{n} a_k F_p(k) = \lim_{p\to\infty} \left(\sum_{k=0}^{n} a_k F_p(k) - F_p(0) \sum_{k=0}^{n} a_k e^k \right) = 0. \qquad (*)$$

Since $j!$ divides all coefficients of j-th derivative of an arbitrary polynomial we obtain for a suitable polynomials P_j with integer coefficients that

$$f_p^{(j)}(x) = \frac{j!}{(p-1)!} P_j(x).$$

Hence we have

$$F_p(0) = \sum_{j=p-1}^{M} f_p^{(j)}(0) = \frac{1}{(p-1)!} \sum_{j=p-1}^{M} j! P_j(0) \equiv P_{p-1}(0) (\mathrm{mod}\, p),$$

Proof

since $f_p(0) = f'_p(0) = \ldots = f_p^{(p-2)}(0) = 0$ and all $\frac{1}{(p-1)!}\sum_{j=p}^{M} j! P_j(0) \in \mathbb{Z}$ and are divisible by p. Similarly, for $f_p^{(i)}(k) = 0$ for $i \in \{1, \ldots, p-1\}$ and $k \in \{1, \ldots, n\}$, thus

$$F_p(k) = \sum_{j=p}^{M} f_p^{(j)}(k) = \frac{1}{(p-1)!}\sum_{j=p}^{M} j! P_j(k) \equiv 0 (\bmod p).$$

Finally,

$$\sum_{k=0}^{n} a_k F_p(k) \equiv a_0 F_p(0) \equiv a_0 P_{p-1}(0) \equiv a_0 (n!)^p \not\equiv 0 (\bmod p).$$

This contradicts with (*), since a sequence of integers that converges to 0 must be constant for all but finitely many terms. This completes the proof of theorem. □