Lesson 4

Dedekind cuts, construction of $\mathbb{R}$ from $\mathbb{Q}$, Consequences of the Axiom of Completeness, Decimals, Extended Real Number System

MATH 411H, FALL 2025

September 15, 2025

Dedekind Cuts

Definition of Dedekind cuts

A Dedekind cut is any subset α of $\mathbb{Q}$ with the following three properties:

(i) $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$.

(ii) If $p \in \alpha$, $q \in \mathbb{Q}$ and $q < p$, then $q \in \alpha$.

(iii) If $p \in \alpha$ then $p < r$ for some $r \in \alpha$.

Remark

- The letters p, q, r will denote rational numbers and α, β, γ will denote Dedekind cuts, which will be simply called cuts.
- Property (iii) simply says that α has no largest member.
- Property (ii) implies two facts which will be freely used:
  - If $p \in \alpha$ and $q \notin \alpha$, then $p < q$.
  - If $r \notin \alpha$ and $r < s$, then $s \notin \alpha$.

The set of real numbers $\mathbb{R}$

Definition of $\mathbb{R}$

We set

$$\mathbb{R} = \{\alpha \subset \mathbb{Q} : \alpha \text{ is a Dedekind cut}\}.$$

Order on $\mathbb{R}$

Define the order on $\mathbb{R}$ by setting

$$\alpha < \beta \quad \text{if} \quad \alpha \subset \beta.$$

Here, α is a proper subset of β, i.e. $\alpha \neq \beta$.

One has to show:

1. if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$,
2. if $\alpha, \beta \in \mathbb{R}$, then only one of the following holds:

$$\alpha < \beta, \quad \text{or} \quad \alpha = \beta, \quad \text{or} \quad \alpha > \beta.$$

Proof

- **Proof of (i)** If $\alpha < \beta$ and $\beta < \gamma$ it is clear that a $\alpha < \gamma$. A proper subset of a proper subset is a proper subset. $\square$
- **Proof of (ii)** It is also clear that at most one of the three relations

$$\alpha < \beta, \quad \text{or} \quad \alpha = \beta, \quad \text{or} \quad \alpha > \beta.$$

  can hold for any pair α, β.
  - To show that at least one holds, assume that the first two fail.
  - Then α is not a subset of β. Hence there is a $p \in \alpha$ with $p \notin \beta$.
  - If $q \in \beta$, it follows that $q < p$ (since $p \notin \beta$), hence $q \in \alpha$, by (i).
  - Thus $\beta \subset \alpha$. Since $\beta \neq \alpha$, we conclude that $\beta < \alpha$. $\square$
- Thus $\mathbb{R}$ is now an ordered set.

Least–upper–bound property

Theorem

The ordered set $\mathbb{R}$ has the least upper bound property.

Proof: To prove this, let $\emptyset \neq A \subseteq \mathbb{R}$, and assume that $\beta \in \mathbb{R}$ is an upper bound of A, i.e. $\alpha < \beta$ for every $\alpha \in A$. Define

$$\gamma = \bigcup_{\alpha \in A} \alpha.$$

In other words, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$. We shall prove that $\gamma \in \mathbb{R}$ and that

$$\gamma = \sup A.$$

- **Proof of property (i):** Since $A \neq \emptyset$, there exists an $\alpha_0 \in A$. This $\alpha_0 \neq \emptyset$ by the property (i). This ensures that $\gamma \neq \emptyset$, since $\alpha_0 \subset \gamma$. Next, $\gamma \subset \beta$ (since $\alpha \subset \beta$ for every $\alpha \in A$), and therefore $\gamma \neq \mathbb{Q}$. Thus γ satisfies property (i). □

Proof

- **Proof of property (ii):** Pick $p \in \gamma$ and $q < p$. We show that $q \in \gamma$.
  - Since $p \in \gamma$, then $p \in \alpha_1$ for some $\alpha_1 \in A$.
  - Since $q < p$, then $q \in \alpha_1$, hence $q \in \gamma$; this proves property (ii). □
- **Proof of property (iii):** Pick $p \in \gamma$. We show that $p < r$ for some $r \in \gamma$.
  - Since $p \in \gamma$, then $p \in \alpha_1$ for some $\alpha_1 \in A$.
  - Choose $r \in \alpha_1$ so that $r > p$, then we see that $r \in \gamma$ (since $\alpha_1 \subset \gamma$), and therefore γ satisfies property (iii).
- We have shown that $\gamma \in \mathbb{R}$. It remain to show that $\gamma = \sup A$.
  - It is clear that $\alpha \leq \gamma$ for every $\alpha \in A$.
  - If $\delta < \gamma$, then there is an $s \in \gamma$ such that $s \notin \delta$. Since $s \in \gamma$, then $s \in \alpha$ for some $\alpha \in A$. Now taking $p \in \delta$ we see that $p < s$, since $s \notin \delta$. But $s \in \alpha$ thus $p \in \alpha$ by property (ii) and consequently $\delta \subset \alpha$.
  - Hence, $\delta < \alpha$, and δ is not an upper bound of A.

This gives the desired result and $\gamma = \sup A$. □

Addition and zero in $\mathbb{R}$

Addition and zero in $\mathbb{R}$

For $\alpha, \beta \in \mathbb{R}$ we define it sum by setting

$$\alpha + \beta = \{r + s : r \in \alpha,\ s \in \beta\}.$$

The neutral element for addition in $\mathbb{R}$ is defined by $0^* = \{u \in \mathbb{Q} : u < 0\}$.

It is easy to check that 0^* is a cut.

Exercise: $\mathbb{R}$ with 0^* is an abelian group satisfying addition axioms (A):

- (A1) if $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$,
- (A2) $x + y = y + x$ for all $x, y \in \mathbb{R}$,
- (A3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$,
- (A4) we have $x + 0^* = x$ for all $x \in \mathbb{R}$,
- (A5) to every $x \in \mathbb{R}$ corresponds an element $(-x) \in \mathbb{R}$ such that

$$x + (-x) = 0^*.$$

Multiplication and one in $\mathbb{R}_+$

Multiplication and one in $\mathbb{R}_+$

We define the set of positive real numbers by

$$\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0^*\}$$

and multiplication in $\mathbb{R}_+$ by setting

$$\alpha\beta = \{p \in \mathbb{Q} : p \leq rs \text{ for some } r \in \alpha,\ s \in \beta,\ r, s > 0\}.$$

The identity element for multiplication in $\mathbb{R}_+$ is defined by

$$1^* = \{q \in \mathbb{Q} : q < 1\}.$$

Exercise

Exercise: $\mathbb{R}$ with 1^* is an abelian group satisfying multiplication axioms (M):

- (M1) if $x, y \in \mathbb{R}_+$, then their product $xy \in \mathbb{R}_+$,
- (M2) $xy = yx$ for all $x, y \in \mathbb{R}_+$,
- (M3) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{R}_+$,
- (M4) we have $1^* \neq 0^*$ and $1^* \cdot x = x$ for all $x \in \mathbb{R}_+$,
- (M5) if $0^* \neq x \in \mathbb{R}_+$ then there is an element $x^{-1} = \frac{1^*}{x} \in \mathbb{R}_+$ such that
$$x \cdot x^{-1} = 1^*.$$

Exercise: $\mathbb{R}_+$ satisfies distributive law (D):

- (D1) $x(y + z) = xy + xz$ holds for all $x, y, z \in \mathbb{R}$.

Multiplication and one in $\mathbb{R}$

Multiplication in $\mathbb{R}$

We complete the definition of multiplication by setting $0^*\alpha = \alpha 0^* = 0^*$, and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) \text{ if } \alpha < 0^* \text{ and } \beta < 0^*, \\ -((-\alpha)\beta) \text{ if } \alpha < 0^* \text{ and } \beta > 0^*, \\ -(\alpha(-\beta)) \text{ if } \alpha > 0^* \text{ and } \beta < 0^*. \end{cases}$$

Exercise

Now $\mathbb{R}$ satisfies the multiplication (M) and the distributive law (D) axioms.

Exercise: $\mathbb{R}$ satisfies ordered field axioms (O):

- (O1) if $x, y, z \in \mathbb{R}$ and $y < z$, then $x + y < x + z$,
- (O2) if $x > 0$ and $y > 0$, then $xy > 0$.

$\mathbb{Q}$ is subfield of $\mathbb{R}$

We associate with each $r \in \mathbb{Q}$ the set

$$r^* = \{p \in \mathbb{Q} : p < r\}.$$

Clearly r^* is a cut and satisfies the following relations:

1. $r^* + s^* = (r+s)^*$,
2. $r^* s^* = (rs)^*$,
3. $r^* < s^* \iff r < s$.

The set of all such cuts will be denoted by

$$\mathbb{Q}^* = \{r^* : r \in \mathbb{Q}\} \subseteq \mathbb{R}.$$

Theorem (Prove it!)

There is a canonical filed isomorphism $\Phi : \mathbb{Q} \to \mathbb{Q}^*$ given by

$$\Phi(r) = r^* \quad \text{for all} \quad r \in \mathbb{Q}.$$

In particular, $\mathbb{Q}$ is a subfield of $\mathbb{R}$ via this identification.

$\mathbb{R}$ is an ordered field satisfying **(AoC)** and contains $\mathbb{Q}$

Theorem (Forget about the previous construction of $\mathbb{R}$!)

There exists a set of real numbers $\mathbb{R}$, which is an ordered field containing $\mathbb{Q}$ and satisfying the axiom of completeness **(AoC)**.

Axiom of completeness (AoC)

Every $\emptyset \neq A \subseteq \mathbb{R}$ that is bounded above has the least–upper–bound.

$\mathbb{R}$ with $+$ is an abelian group satisfying addition axioms (A):

- (A1) if $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$,
- (A2) $x + y = y + x$ for all $x, y \in \mathbb{R}$,
- (A3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$,
- (A4) $\mathbb{R}$ contains the element 0 such that $x + 0 = x$ for all $x \in \mathbb{R}$,
- (A5) to every $x \in \mathbb{R}$ corresponds an element $(-x) \in \mathbb{R}$ such that
$$x + (-x) = 0.$$

Proof

$\mathbb{R}$ with $\cdot$ is an abelian group satisfying multiplication axioms (M):

- (M1) if $x, y \in \mathbb{R}$, then their product $xy \in \mathbb{R}$,
- (M2) $xy = yx$ for all $x, y \in \mathbb{R}$,
- (M3) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{R}$,
- (M4) $\mathbb{R}$ contains the element $1 \neq 0$ such that $1 \cdot x = x$ for all $x \in \mathbb{R}$,
- (M5) if $0 \neq x \in \mathbb{R}$ then there is an element $x^{-1} = \frac{1}{x} \in \mathbb{R}$ such that

$$x \cdot x^{-1} = 1.$$

$\mathbb{R}$ with $+$ and $\cdot$ satisfies distributive law (D):

- (D1) $x(y + z) = xy + xz$ holds for all $x, y, z \in \mathbb{R}$.

$\mathbb{R}$ with $<$ satisfies ordered field axioms (O):

- (O1) if $x, y, z \in \mathbb{R}$ and $y < z$, then $x + y < x + z$,
- (O2) if $x > 0$ and $y > 0$, then $xy > 0$.

Nested interval property

Theorem

Nested interval property For each $n \in \mathbb{N}$, assume we are given a closed interval

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} \ : \ a_n \leq x \leq b_n\}.$$

Assume also that $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$. Then the resulting nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

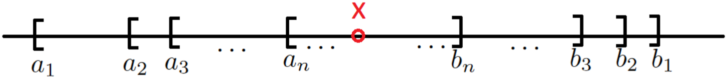
has a nonempty intersection, that is

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset.$$

Proof 1/2

Proof: Using **(AoC)** we will produce $x \in \mathbb{R}$ so that $x \in I_n$ for every $n \in \mathbb{N}$. Then

$$\bigcap_{n \in \mathbb{N}} I_n \supset \{x\} \neq \emptyset.$$



Consider the set $A = \{a_n \ : \ n \in \mathbb{N}\}$ of all left-hand endpoints of the intervals I_n. Because the intervals are nested one sees that every b_n serves as an upper bound for A. Thus by the **(AoC)** we are allowed to write

$$x = \sup A \in \mathbb{R}.$$

The proof will be complete if we show that $x \in I_n$ for all $n \in \mathbb{N}$.

Proof 2/2

Since x is an upper bound for A thus

$$a_n \leq x \qquad \text{for all} \qquad n \in \mathbb{N}.$$

The fact that b_n is an upper bound for A and that x is the least upper bound implies

$$x \leq b_n \qquad \text{for all} \qquad n \in \mathbb{N}.$$

Thus

$$a_n \leq x \leq b_n$$

for all $n \in \mathbb{N}$ hence $x \in I_n$ for all $n \in \mathbb{N}$ and consequently

$$x \in \bigcap_{n \in \mathbb{N}} I_n.$$

□

Archimedian property of $\mathbb{R}$

Archimedian property

(i) Given any number $x, z \in \mathbb{R}$ with $z > 0$ there exists $n \in \mathbb{N}$ satisfying

$$nz > x.$$

(ii) Given any real number $y > 0$ there exists an $n \in \mathbb{N}$ satisfying

$$\frac{1}{n} < y.$$

Proof

Proof. Note that (i) implies (ii) by letting $x = \frac{1}{y}$ and $z = 1$. It suffices to prove (i). Without loss of generality we can assume that $x > 0$ and consider

$$A = \{nz \ : \ n \in \mathbb{N}\}.$$

Suppose for a contradiction that A is bounded, i.e. there is $y \geq 0$ such that $nz \leq y$ for any $n \in \mathbb{N}$. This means that y is an upper bound for A. By the **(AoC)**:

$$\alpha = \sup A \in \mathbb{R}.$$

Since $z > 0$, $\alpha - z < \alpha$ and $\alpha - z$ is not upper bound of A. Thus we find $m \in \mathbb{N}$ such that

$$\alpha - z < mz \iff \alpha < (m+1)z.$$

This is contradiction since α is the supremum of A. □

$\mathbb{Q}$ is dense in $\mathbb{R}$

Theorem ($\mathbb{Q}$ is dense in $\mathbb{R}$)

If $x, y \in \mathbb{R}$ and $x < y$ then there is $p \in \mathbb{Q}$ such that $x < p < y$.

Proof. Since $x < y$, by Archimedian property there is $n \in \mathbb{N}$ such that

$$n(y - x) > 1.$$

- Then, we apply Archimedian property to find $m_1, m_2 \in \mathbb{Z}$ such that $m_1 > nx$ and $m_2 > -nx$. Then $-m_2 < nx < m_1$.
- Hence there is an integer m with $-m_2 \leq m \leq m_1$ such that

$$m - 1 \leq nx < m.$$

- We combine these inequalities to get

$$nx < m \leq nx + 1 < ny, \qquad \text{so} \qquad x < p = \frac{m}{n} < y.$$

□

n-th root of a real number

Theorem

For every real $x > 0$ and $n \in \mathbb{N}$ there is a unique real number $y > 0$ so that

$$y^n = x.$$

The number $y > 0$ is called the ***n*-th root of** x and we will write $y = \sqrt[n]{x}$.

Proof: Uniqueness. The fact that there exists at most one such y is clear, since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$.

Identity $b^n - a^n$

In the proof (in the existence part), we will use the following identity

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-2}b + a^{n-1}),$$

which holds for all $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$.

Proof: 1/3

- **Proof: Existence.** Let

$$E = \{t > 0 \ : \ t^n < x\}.$$

- If $t = \frac{x}{x+1}$, then $0 \le t < 1$ hence

$$t^n \le t < x$$

  thus $t \in E$ and $E \neq \emptyset$.
- If $t > x + 1$, then $t^n > t > x$, so that $t \notin E$. Thus $1 + x$ is an upper bound of E.
- By the **(AoC)** we may write $y = \sup E \in \mathbb{R}$. We will show that

$$y^n = x.$$

- It suffices to show that $y^n < x$ and $y^n > x$ cannot hold.

Proof: 2/3. Case $y^n < x$.

- The identity
$$b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \ldots + a^{n-2}b + a^{n-1})$$
gives
$$b^n - a^n < (b-a)nb^{n-1}$$
if $0 < a < b$.
- Assume $y^n < x$. Choose $0 < h < 1$ so that
$$h < \frac{x - y^n}{n(y+1)^{n-1}}.$$
- Put $a = y$, $b = y + h$. Then
$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$
- Thus $(y+h)^n < x$ and $y + h \in E$. Since $y + h > y$ this contradicts the fact that y is an upper bound of E.

Proof: 3/3. Case $y^n > x$.

- Assume that $y^n > x$ and set
$$k = \frac{y^n - x}{ny^{n-1}}.$$
- Then $0 < k < y$. If $t \geq y - k$ we conclude
$$y^n - t^n \leq y^n - (y-k)^n < kny^{n-1} = y^n - x.$$
- Thus $t^n > x$ and $t \notin E$. It follows that $y - k$ is an upper bound of E. But
$$y - k < y,$$
which contradicts the fact that y is the least upper bound of E.
- Hence
$$y^n = x.$$

Corollary

Corollary

If $a, b > 0$ are real numbers and $n \in \mathbb{N}$, then

$$(ab)^{1/n} = a^{1/n} b^{1/n}$$

It is a consequence of the uniqueness property in the previous theorem.

Exercise: x^y for $x, y \in \mathbb{R}$

Fix $b > 1$.

- If $m, n, p, q \in \mathbb{Z}$, $n, q > 0$ and $r = \frac{m}{n} = \frac{p}{q}$, then
$$(b^m)^{\frac{1}{n}} = (b^p)^{\frac{1}{q}}.$$
- Hence, it makes sense to define $b^r = (b^m)^{\frac{1}{n}}$.
- If $r, s \in \mathbb{Q}$, then
$$b^{r+s} = b^r b^s.$$
- If $x \in \mathbb{R}$ define
$$B(x) = \{b^t \ : \ t \in \mathbb{Q},\ t \leq x\}.$$
- Then $b^r = \sup B(r)$ when $r \in \mathbb{Q}$. Hence, it makes sense to define
$$b^x = \sup B(x)$$
for every $x \in \mathbb{R}$.

Decimals 1/2

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \leq x$.

Remark

Note that the existence of n_0 follows from the Archimedean property. Why?

Then, we define n_1 to be the largest integer such that

$$n_0 + \frac{n_1}{10} \leq x.$$

then, having n_0, n_1, we define n_2 to be the largest integer such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{100} \leq x.$$

We continue this procedure...

Decimals 2/2

Having chosen

$$n_0, n_1, \dots, n_{k-1}$$

let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^k} \leq x.$$

Let

$$E = \{n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^k} \ : \ k \in \mathbb{N}_0\}.$$

Then one can show that $x = \sup E$.

Decimal system - example

Example

Write down $0,25$ in the form $\frac{n}{m}$.

Solution. We write

$$0,25 = \frac{2}{10} + \frac{5}{100} = \frac{20}{100} + \frac{5}{100} = \frac{25}{100} = \frac{1}{4}.$$

□

Decimal system - example

Example

Write down $x = 0,101010101\dots$ in the form $\frac{n}{m}$.

Solution. Note that

$$10x = 10,10101010\dots,$$

hence

$$10x = 10 + x$$

$$9x = 10 \iff x = \frac{10}{9}.$$

□

The extended real number system

The extended real number system

The extended real number system consists of real numbers $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$.

We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty$$

for all $x \in \mathbb{R}$.

Example

If $E \subseteq \mathbb{R}$, $E \neq \emptyset$ but not bounded then

$$\sup E = +\infty.$$

Properties of the extended real number system

Properties

If $x \in \mathbb{R}$, then

1. (A) $x + \infty = \infty$, $x - \infty = -\infty$, $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$,
2. (B) if $x > 0$, then $x(+\infty) = +\infty$, $x(-\infty) = -\infty$,
3. (C) if $x < 0$, then $x(+\infty) = -\infty$, $x(-\infty) = +\infty$.