Lecture 6

Subsequences and Cauchy Sequences
Monotone Convergence Theorem and Bolzano–Weierstrass Theorem
Cauchy Completeness and Complex field

MATH 411H, FALL 2025

September 22, 2025

Subsequences

Definition

Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, and $n_1 < n_2 < \ldots < n_k < \ldots$ be an increasing sequence of positive integers. Then the sequence

$$(a_{n_1}, a_{n_2}, \ldots, a_{n_k}, \ldots)$$

is called **a subsequence** of $(a_n)_{n\in\mathbb{N}}$ and is denoted by $(a_{n_k})_{k\in\mathbb{N}}$.

Example

Let $(a_n)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots, \right)$, then $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right)$ and $\left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots\right)$ are subsequences of $(a_n)_{n\in\mathbb{N}}$. The sequences

$$\left(\frac{1}{10}, \frac{1}{2}, \frac{1}{100}, \ldots\right) \quad \text{and} \quad (1, 1, \ldots) \quad \text{are NOT!.}$$

Limit of a subsequence

Theorem

Subsequences of a convergent sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converge to the same limit as the original sequence.

Proof. Assume $\lim_{n\to\infty} a_n = a$ and let $(a_{n_k})_{k\in\mathbb{N}}$ be a subsequence. Given $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ so that

$$n \geq N_\varepsilon \qquad \text{implies} \qquad |a_n - a| < \varepsilon.$$

Because $n_k \geq k$ for all $k \in \mathbb{N}$, the same N_ε will suffice for the subsequence, that is

$$|a_{n_k} - a| < \varepsilon \qquad \text{whenever} \qquad k \geq N_\varepsilon.$$

□

Cauchy sequences

Cauchy sequences

A sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is called **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that whenever $m, n \geq N_\varepsilon$ it follows

$$|a_n - a_m| < \varepsilon.$$

Convergent sequences

Recall that a sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges to $a \in \mathbb{R}$ if for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that whenever $n \geq N_\varepsilon$ if follows

$$|a_n - a| < \varepsilon.$$

Convergent sequences are Cauchy

Theorem

Every convergent sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is a Cauchy sequence.

Proof. Let $\varepsilon > 0$ be given. If

$$\lim_{n\to\infty} x_n = x,$$

then there is $N_\varepsilon \in \mathbb{N}$ so that $n \geq N_\varepsilon$ implies

$$|x_n - x| < \frac{\varepsilon}{2}.$$

Thus for $n, m \geq N_\varepsilon$ we obtain

$$|x_m - x_n| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is completed. □

Cauchy sequences are bounded

Lemma

Cauchy sequences $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ are bounded.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be Cauchy. Given $\varepsilon = 1$ there is $N \in \mathbb{N}$ so that if $n, m \geq N$ then $|x_n - x_m| < 1$. Thus

$$|x_n| \leq |x_N| + 1.$$

Taking

$$M = \max\{|x_1|, |x_2|, \dots, |x_N|, |x_N| + 1\}$$

we conclude $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

Cauchy sequences and converging subsequences

Theorem

Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a Cauchy sequence. Suppose that there is $(n_k)_{k\in\mathbb{N}}$ so that $\lim_{k\to\infty} x_{n_k} = x$. Then $\lim_{n\to\infty} x_n = x$.

Proof. Assume that $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is Cauchy and there is $(n_k)_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} x_{n_k} = x \in \mathbb{F} \qquad (*).$$

Let $\varepsilon > 0$ be given. Then there is $N_\varepsilon \in \mathbb{N}$ so that $n, m \geq N_\varepsilon$ implies $|x_n - x_m| < \frac{\varepsilon}{2}$. By (*) we can choose $n_k \in \mathbb{N}$ so that $n_k \geq N_\varepsilon$ and

$$|x_{n_k} - x| < \frac{\varepsilon}{2}.$$

Then for $n \geq N_\varepsilon$ and the triangle inequality

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Increasing and decreasing sequences

Increasing and decreasing sequences

Let $\mathbb{R}$ be an ordered field. A sequence of real numbers $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is

- **increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$;
- **decreasing** if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Monotone sequence

A sequence is **monotone** if it is either increasing or decreasing.

Example

- $\left(3 + \frac{1}{n}\right)_{n\in\mathbb{N}}$ is decreasing, so it is monotone.
- $\left(n^3\right)_{n\in\mathbb{N}}$ is increasing, so it is monotone.
- $((-1)^n)_{n\in\mathbb{N}}$ is neither increasing nor decreasing, so it is not monotone.

Monotone convergence theorem

Monotone convergence theorem (MCT)

If a sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is monotone and bounded then it converges.

Proof. Assume that $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded. Consider the set

$$E = \{x_n \colon n \in \mathbb{N}\} \subseteq \mathbb{R},$$

which is nonempty and bounded. Let $x = \sup E \in \mathbb{R}$, which exists by the axiom of completeness (AoC). We will show that $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$ and note that there exists $N_\varepsilon \in \mathbb{N}$ so that

$$x - \varepsilon < x_{N_\varepsilon} \leq x.$$

But $(x_n)_{n\in\mathbb{N}}$ is increasing thus for any $n \geq N_\varepsilon$ one has

$$x - \varepsilon < x_{N_\varepsilon} \leq x_n \leq x < x + \varepsilon.$$

Hence $|x_n - x| < \varepsilon$ for all $n \geq N_\varepsilon$, which shows that $\lim_{n\to\infty} x_n = x$. □

Bolzano–Weierstrass theorem

Theorem

Every bounded sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ *contains a convergent subsequence.*

Proof. Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be bounded. Then there is $M > 0$ such that

$$|a_n| \leq M \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus $a_n \in [-M, M]$ for all $n \in \mathbb{N}$.

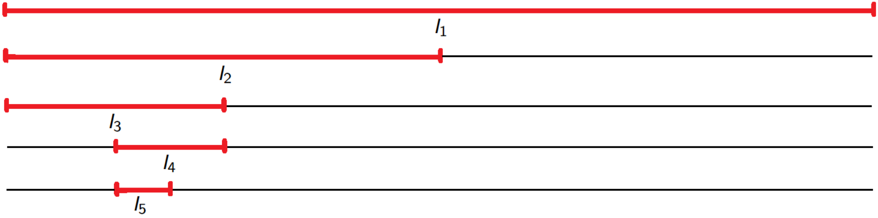
- **Step 1.** Divide $[-M, M]$ into two closed intervals $[-M, 0]$, $[0, M]$. We can assume (wlog) that $I_1 = [0, M]$ contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$. Moreover, the length of I_1 is M.
- **Step 2.** Divide I_1 into two closed intervals of the same length and select the one which contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$. Call it $I_2 \subset I_1$ and note that has length $\frac{M}{2}$.

Proof: 1/2

- **Step 3.** Proceeding inductively as above we obtain a sequence of decreasing closed intervals

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset \dots$$

where each I_k contains infinitely many elements of $(a_n)_{n\in\mathbb{N}}$ and has length $\frac{M}{2^{k-1}}$.



Proof: 2/2

- **Step 4.** By the nested intervals property $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$. In fact,

$$\bigcap_{k=1}^{\infty} I_k = \{x\} \quad \text{for some} \quad x \in \mathbb{R} \quad \text{why?}.$$

  Now for each $k \in \mathbb{N}$ select an element $a_{n_k} \in I_k$ so that

$$n_1 < n_2 < \ldots < n_k < \ldots$$

  where a_{n_1} is any element of I_1.

- **Step 5.** Let $\varepsilon > 0$ and choose $N_\varepsilon \in \mathbb{N}$ so that

$$\frac{M}{2^{k-1}} \leq \frac{2M}{k} < \varepsilon \quad \text{for} \quad k \geq N_\varepsilon.$$

  Then for every $k \geq N_\varepsilon$ we have

$$|a_{n_k} - x| \leq \frac{M}{2^{k-1}} < \varepsilon,$$

  thus $\lim_{n\to\infty} a_{n_k} = x$. □

Bolzano–Weierstrass theorem implies Cauchy completeness

Example

Let us consider a sequence $a_n = (-1)^n$. It in **NOT** convergent, but the subsequence $(-1)^{2n} = 1$ converges to 1.

Theorem (Cauchy completeness of $\mathbb{R}$)

A sequence $(x_n)_{n\in\mathbb{N}}$ converges iff it is a Cauchy sequence.

Proof: The implication ($\Longrightarrow$) has already been proved. For the reverse implication ($\Longleftarrow$) assume that $(x_n)_{n\in\mathbb{N}}$ is Cauchy, thus it is bounded. By the **Bolzano–Weierstrass** theorem there is $(n_k)_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} x_{n_k} = x \quad \text{for some} \quad x \in \mathbb{R}.$$

But Cauchy sequences with converging subsequences converge, i.e.

$$\lim_{n\to\infty} x_n = x.$$

This completes the proof. □

Complex numbers

Definition (Complex numbers)

A **complex number** is an ordered pair $(a, b) \in \mathbb{R} \times \mathbb{R}$.

Definition (Addition and multiplication of complex numbers)

For two complex numbers $x = (a, b), y = (c, d) \in \mathbb{R} \times \mathbb{R}$ we define

- **addition** $+$ by setting
$$x + y = (a + c, b + d),$$
- **multiplication** $\cdot$ by setting
$$x \cdot y = (ac - bd, ad + bc).$$

Complex field

Theorem

These operations addition $+$ and multilpication $\cdot$ turn the set of all complex numbers into a field with $(0,0)$ and $(1,0)$ playing, respectively, the role of 0 and 1. This field will be denoted by $\mathbb{C}$.

Proof. We have to verify the field axioms.

Addition axioms (A)

- (A1) if $x, y \in \mathbb{C}$, then $x + y \in \mathbb{C}$,
- (A2) $x + y = y + x$ for all $x, y \in \mathbb{C}$,
- (A3) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{C}$,
- (A4) $\mathbb{C}$ contains the element 0 such that $x + 0 = x$ for all $x \in \mathbb{C}$,
- (A5) to every $x \in \mathbb{C}$ corresponds an element $(-x) \in \mathbb{C}$ such that

$$x + (-x) = 0.$$

Proof

Multiplication axioms (M)

- (M1) if $x, y \in \mathbb{C}$, then their product $xy \in \mathbb{C}$,
- (M2) $xy = yx$ for all $x, y \in \mathbb{C}$,
- (M3) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{C}$,
- (M4) $\mathbb{C}$ contains the element $1 \neq 0$ such that $1 \cdot x = x$ for all $x \in \mathbb{C}$,
- (M5) if $0 \neq x \in \mathbb{C}$ then there is an element $x^{-1} = \frac{1}{x} \in \mathbb{C}$ such that

$$x \cdot x^{-1} = 1.$$

Distributive law (D)

- (D1) $x(y+z) = xy + xz$ holds for all $x, y, z \in \mathbb{C}$.

Let $x = (a, b), y = (c, d), z = (e, f)$. We will use the field structure of $\mathbb{R}$.

- **Proof of (A1).** By the definition of addition

$$x + y = (a, b) + (c, d) = (a + c, b + d) \in \mathbb{C}.$$

Proof

- **Proof of (A2).**

$$x + y = (a + c, b + d) = (c + a) + (d + b) = y + x.$$

- **Proof of (A3).**

$$\begin{aligned}(x + y) + z &= (a + c, b + d) + (e, f) \\ &= (a + c + e, b + d + f) \\ &= (a, b) + (c + e, d + f) = x + (y + z).\end{aligned}$$

- **Proof of (A4).**

$$x + 0 = (a, b) + (0, 0) = (a, b) = x.$$

- **Proof of (A5).** Set $-x = (-a, -b)$ and note that

$$x + (-x) = (a - a, b - b) = (0, 0) = 0.$$

Proof

- **Proof of (M1).** By the definition of multiplication

$$x \cdot y = (a, b) \cdot (c, d) = (ac - bd, ad + bc) \in \mathbb{C}.$$

- **Proof of (M2).**

$$x \cdot y = (ac - bd, ad + bc) = (ca - db, da + cb) = y \cdot x.$$

- **Proof of (M3).**

$$\begin{aligned}(x \cdot y) \cdot z &= (ac - bd, ad + bc) \cdot (e, f) \\ &= (ace - bde - adf - bcf, acf - bdf + ade + bce) \\ &= (a, b) \cdot (ce - df, cf + de) = x \cdot (y \cdot z).\end{aligned}$$

- **Proof of (M4).**

$$1 \cdot x = (1, 0) \cdot (a, b) = (a, b) = x.$$

Proof

- **Proof of (M5).** If $x \neq 0$ then $(a, b) \neq (0, 0)$, which means that at least one of the real numbers a, b is different from 0. Hence $a^2 + b^2 > 0$ and we define

$$\frac{1}{x} = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right).$$

Then

$$x \cdot \frac{1}{x} = (a, b) \cdot \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1, 0).$$

- **Proof of (D1).**

$$\begin{aligned} x \cdot (y + z) &= (a, b) \cdot (c + e, d + f) \\ &= (ac + ae - bd - bf, ad + af + bc + be) \\ &= (ac - bd, ad + bc) + (ae - bf, af + be) \\ &= x \cdot y + x \cdot z. \end{aligned}$$

This completes the proof that $\mathbb{C}$ is a field. □

Imaginary number i

Remark

For any $a, b \in \mathbb{R}$ we have

$$(a, 0) + (b, 0) = (a + b, 0) \quad \text{and} \quad (a, 0) \cdot (b, 0) = (ab, 0).$$

- The complex numbers from the set $\{(a, 0) : a \in \mathbb{R}\}$ have the same arithmetic properties as the corresponding real numbers $\mathbb{R}$.
- We can therefore identify $(a, 0)$ with a. This identification gives us the real field $\mathbb{R}$ as a subfield of the complex field $\mathbb{C}$.
- We have defined the complex numbers $\mathbb{C}$ without any reference to the mysterious square root of -1. We now show that the notation (a, b) is equivalent to the more customary $a + bi$.

Definition

We define the **imaginary number** by setting $i = (0, 1)$.

Equivalent definition of $\mathbb{C}$

Theorem

One has that $i^2 = -1$.

Proof.

Note that $i^2 = (0,1) \cdot (0,1) = (-1,0)$. □

Theorem

We also have

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

Proof.

It suffices to note that

$$\begin{aligned} a + ib &= (a,0) + (0,1) \cdot (b,0) \\ &= (a,0) + (0,b) = (a,b). \end{aligned}$$

□

Conjugate, real and imaginary parts

Definition

If $z \in \mathbb{C}$ and $z = a + ib$ for some $a, b \in \mathbb{R}$ then the complex number

$$\overline{z} = a - ib$$

is called the **conjugate** of z. The numbers a and b are the **real part** and **imaginary part** of z respectively. We shall write

$$a = \Re(z) = \mathrm{Re}(z) \quad \text{and} \quad b = \Im(z) = \mathrm{Im}(z).$$

Theorem

If $z, w \in \mathbb{C}$ then

(i) $\overline{z + w} = \overline{z} + \overline{w}$.

(ii) $\overline{zw} = \overline{z} \cdot \overline{w}$.

(iii) $z + \overline{z} = 2\mathrm{Re}(z)$ *and* $z - \overline{z} = 2i\mathrm{Im}(z)$.

(iv) $z\overline{z}$ *is a positive real number except when* $z = 0$.

Proof

Proof. Let $z = a + ib$ and $w = c + id$.

- **Proof of (i).** Note that

$$\overline{z+w} = \overline{(a+c) + i(b+d)} = (a+c) - i(b+d) = \overline{z} + \overline{w}.$$

- **Proof of (ii).** Note that

$$\overline{z \cdot w} = (ac - bd) - i(ad + bc) \quad \text{and}$$
$$\overline{z} \cdot \overline{w} = (a - ib)(c - id) = (ac - bd) - i(ad + bc).$$

- **Proof of (iii).** We have

$$z + \overline{z} = (a + ib) + (a - ib) = 2a = 2\mathrm{Re}(z),$$
$$z - \overline{z} = (a + ib) - (a - ib) = 2ib = 2i\mathrm{Im}(z).$$

- **Proof of (iv).** We have $z \cdot \overline{z} = (a + ib)(a - ib) = a^2 + b^2 > 0$ if and only if $z \neq 0$. □

Absolute value on $\mathbb{C}$

Definition

If $z \in \mathbb{C}$ its **absolute value** $|z|$ is defined by setting

$$|z| = \sqrt{z \cdot \overline{z}}.$$

Remark

This absolute value exists and is unique. Moreover, it coincides with the absolute value from $\mathbb{R}$. If $x \in \mathbb{R}$ then $\overline{x} = x$ hence $|x| = \sqrt{x \cdot \overline{x}} = \sqrt{x^2}$. Thus

$$|x| = \begin{cases} x & \text{if } x \leq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Properties of the absolute value on $\mathbb{C}$

Theorem

If $z, w \in \mathbb{C}$ then

(i) $|z| > 0$ *if and only if* $z \neq 0$, *and* $|0| = 0$.

(ii) $|\overline{z}| = |z|$.

(iii) $|zw| = |z||w|$.

(iv) $|\mathrm{Re}(z)| \leq |z|$ *and* $|\mathrm{Im}(z)| \leq |z|$

(v) $|z + w| \leq |z| + |w|$.

Proof. Let $z = a + ib$ and $w = c + id$.

- **Proof of (i).** From the previous theorem we have

$$|z|^2 = z \cdot \overline{z} = (a + ib)(a - ib) = a^2 + b^2 > 0,$$

which gives the desired claim.

Proof

- **Proof of (ii).** Note that $|z|^2 = a^2 + b^2 = |\bar{z}|^2$.
- **Proof of (iii).** Note that

$$|z \cdot w| = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2|w|^2.$$

- **Proof of (iv).** We have

$$|\mathrm{Re}(z)| = |a| \leq \sqrt{a^2 + b^2} = |z|, \quad \text{and} \quad |\mathrm{Im}(z)| = |b| \leq \sqrt{a^2 + b^2} = |z|.$$

- **Proof of (v).** Note that $\bar{z}w$ is the conjugate of $z\overline{w}$ so that $z\overline{w} + \bar{z}w = 2\mathrm{Re}(z\overline{w})$. Hence

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \overline{w}) = z\bar{z} + z\overline{w} + \bar{z}w + w\overline{w} \\ &= |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|\mathrm{Re}(z\overline{w})| + |w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2. \end{aligned}$$

The proof of the theorem is completed. □

Convergence in $\mathbb{C}$

Definition

We say that a sequence of complex numbers $(z_n)_{n\in\mathbb{N}} \subseteq \mathbb{C}$ converges to $z \in \mathbb{C}$ if and only if

$$\lim_{n\to\infty} |z_n - z| = 0.$$

We write

$$\lim_{n\to\infty} z_n = z \quad \text{if and only if} \quad \lim_{n\to\infty} |z_n - z| = 0.$$

This is also equivalent to say that for every $\varepsilon > 0$ there exists an integer $N_\varepsilon \in \mathbb{N}$ such that if $n \geq N_\varepsilon$ then

$$|z_n - z| < \varepsilon.$$