Lecture 7

More about sequences
Classical inequalities in analysis

MATH 411H, FALL 2025

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Tools

Theorem (Squeeze Theorem)

Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be sequences such that $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L \in \mathbb{R}$, then $\lim_{n\to\infty} y_n = L$.

Corollary (Monotone convergence theorem (MCT))

Every bounded and monotonic sequence in $\mathbb{R}$ converges to some $x \in \mathbb{R}$.

Binomial theorem

For every $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ one has

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, for all $n \in \mathbb{N}$ and $0! = 1$.

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

Theorem

Theorem

- (a) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
- (b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$
- (c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
- (d) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
- (e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof of (a): Take $\varepsilon > 0$ be arbitrary, but fixed. Then

$$n > \left(\frac{1}{\varepsilon}\right)^{1/p},$$

which is possible by the Archimedean property.

Proof of (b):

Proof of (b): If $p > 1$ set $x_n = \sqrt[n]{p} - 1$, then $x_n > 0$ and by **Bernoulli's inequality**

$$1 + nx_n \leq (1 + x_n)^n = p,$$

so that

$$0 < x_n \leq \frac{p-1}{n}.$$

But

$$\lim_{n\to\infty} \frac{p-1}{n} = 0,$$

thus by the squeeze theorem we conclude

$$\lim_{n\to\infty} x_n = 0$$

as desired.

Proof of (c):

Proof of (c): Set $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and by the **binomial theorem**

$$n = (1 + x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \leq x_n \leq \left(\frac{2}{n-1}\right)^{1/2} \quad \text{for} \quad n \geq 2.$$

But

$$\lim_{n\to\infty} \left(\frac{2}{n-1}\right)^{1/2} = 0.$$

Thus by the squeeze theorem

$$\lim_{n\to\infty} x_n = 0$$

as desired. □

Proof of (d) and (e):

Proof of (d): Let $k \in \mathbb{N}$ so that $k > \alpha$. For $n > 2k$ by the **binomial theorem**

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)\dots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!},$$

since $n \geq \frac{n}{2}, n-1 \geq \frac{n}{2}, \dots, n-k+1 \geq \frac{n}{2}$. Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} \quad \text{for} \quad n > 2k.$$

Since $\alpha - k < 0$ thus $\lim_{n\to\infty} n^{\alpha-k} = 0$ by (a) and by the squeeze theorem $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$. □

Proof of (e): Take $\alpha = 0$ in (d) and observe that if $0 < x < 1$ then the sequence $x_n = x^n$ is decreasing and bounded. Thus $\lim_{n\to\infty} x_n = 0$. □

Proposition

Proposition

If $a > 0$ and $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} a^{x_n} = a^{x_0}$.

Proof. It suffices to prove that $\lim_{n\to\infty} a^{x_n} = 1$ if $\lim_{n\to\infty} x_n = 0$. Assume $a > 1$. By the previous theorem we know that

$$\lim_{n\to\infty} a^{1/n} = \lim_{n\to\infty} a^{-1/n} = 1.$$

Thus for any $\varepsilon > 0$ there is $M_\varepsilon \in \mathbb{N}$ such that for any $m \geq M_\varepsilon$

$$1 - \varepsilon < a^{-1/m} < a^{1/m} < 1 + \varepsilon.$$

Now since $\lim_{n\to\infty} x_n = 0$ we find $N_{m,\varepsilon} \in \mathbb{N}$ so that for $n \geq N_{\varepsilon,m}$

$$|x_n| < \frac{1}{m} \iff -\frac{1}{m} < x_n < \frac{1}{m}.$$

Proof of Proposition

Thus

$$1-\varepsilon < a^{-1/m} < a^{x_n} < a^{1/m} < 1+\varepsilon$$

which proves $|a^{x_n} - 1| < \varepsilon$ for any $n \geq N_{m,\varepsilon}$ proving that

$$\lim_{n\to\infty} a^{x_n} = 1.$$

If $0 < a < 1$ we note that

$$\lim_{n\to\infty} a^{x_n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{a}\right)^{x_n}}$$

and this completes the proof of the proposition. □

Geometric and arithmetic means

Let $a_1, a_2, \ldots, a_n \geq 0$ be given.

Arithmetic mean

We define **arithmetic mean** of $a_1, a_2, \ldots, a_n$ by

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

Geometric mean

We define **geometric mean** of $a_1, a_2, \ldots, a_n$ by

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}.$$

Geometric Mean vs Arithmetic Mean

Theorem

For any $n \in \mathbb{N}$ we have
$$G_n \leq A_n.$$

Proof. For $n = 2$ observe that
$$(a-b)^2 \geq 0,$$
since
$$a^2 - 2ab + b^2 \geq 0 \quad \Longleftrightarrow \quad ab \leq \frac{a^2+b^2}{2}.$$
Taking $a = \sqrt{a_1}$ and $b = \sqrt{a_2}$ we obtain
$$A_2 = \frac{a_1 + a_2}{2} = \frac{(\sqrt{a_1})^2 + (\sqrt{a_2})^2}{2} \geq \sqrt{a_1 a_2} = G_2.$$

Cases $n = 4$ and $n = 8$ suggest induction

Case $n = 4$. Note that

$$A_4 = \frac{a_1 + a_2 + a_3 + a_4}{4} = \frac{1}{2}\left(\frac{a_1 + a_2}{2} + \frac{a_3 + a_4}{2}\right)$$

$$\underbrace{\geq}_{A_2 \geq G_2} \left((a_1 a_2)^{1/2} (a_3 a_4)^{1/2}\right)^{1/2} = (a_1 a_2 a_3 a_4)^{1/4} = G_4.$$

Case $n = 8$. Let us use $A_4 \geq G_4$ and $A_2 \geq G_2$ to prove $A_8 \geq G_8$.

$$A_8 = \frac{a_1 + \ldots + a_8}{8} = \frac{1}{2}\left(\frac{a_1 + \ldots + a_4}{4} + \frac{a_5 + \ldots + a_8}{4}\right)$$

$$\underbrace{\geq}_{A_2 \geq G_2} \left(\frac{a_1 + \ldots + a_4}{4} \frac{a_5 + \ldots + a_8}{4}\right)^{1/2}$$

$$\underbrace{\geq}_{A_4 \geq G_4} \left((a_1 \ldots a_4)^{1/4} (a_5 \ldots a_8)^{1/4}\right)^{1/2} = (a_1 \ldots a_8)^{1/8} = G_8.$$

Claim and base step

We first use induction to prove

$$A_{2^n} \geq G_{2^n}$$

for all $n \in \mathbb{N}$.

Base step. For $n = 2$ the inequality is true as

$$A_2 = \frac{a_1 + a_2}{2} \geq (a_1 a_2)^{1/2} = G_2.$$

Inductive step

Let $P(n)$ be the statement that $A_{2^n} \geq G_{2^n}$ holds for some $n \in \mathbb{N}$.

Inductive step. Now we prove that $P(n) \implies P(n+1)$. Indeed,

$$\begin{aligned} A_{2^{n+1}} &= \frac{a_1 + \ldots + a_{2^{n+1}}}{2^{n+1}} = \frac{1}{2}\left(\frac{a_1 + \ldots + a_{2^n}}{2^n} + \frac{a_{2^n+1} + \ldots + a_{2^{n+1}}}{2^n}\right) \\ &\underbrace{\geq}_{A_2 \geq G_2} \left(\frac{a_1 + \ldots + a_{2^n}}{2^n}\frac{a_{2^n+1} + \ldots + a_{2^{n+1}}}{2^n}\right)^{1/2} \\ &\underbrace{\geq}_{A_{2^n} \geq G_{2^n}} \left((a_1 \ldots a_{2^n})^{1/2^n}(a_{2^n+1} \ldots a_{2^{n+1}})^{1/2^n}\right)^{1/2} \\ &= (a_1 \ldots a_{2^{n+1}})^{1/2^{n+1}} = G_{2^{n+1}}. \end{aligned}$$

Proof of GM vs AM inequality

Now we have to show that

$$A_n \geq G_n$$

for all $n \in \mathbb{N}$.

We first observe that the following downwards induction holds.

Let $Q(n)$ be the statement that

$$A_n \geq G_n$$

holds for some $n \in \mathbb{N}$. Then

$$Q(n-1)$$

is also true.

This will follow from the so-called **bootstrap phenomenon**.

Bootstrap phenomenon

Note that (by $A_n \geq G_n$ with $a_1, a_2, \ldots, a_{n-1}, a_n = A_{n-1}$) one has

$$\frac{a_1 + \ldots + a_{n-1} + A_{n-1}}{n} \geq (a_1 \cdot \ldots \cdot a_{n-1} \cdot A_{n-1})^{1/n}.$$

But

$$\frac{a_1 + \ldots + a_{n-1} + A_{n-1}}{n} = \frac{(n-1)A_{n-1} + A_{n-1}}{n} = A_{n-1}.$$

Thus we have shown

Bootstrapping inequality

$$A_{n-1} \geq (a_1 \cdot \ldots \cdot a_{n-1})^{1/n} A_{n-1}^{1/n}.$$

Hence

$$A_{n-1}^{1-1/n} \geq (a_1 \cdot \ldots \cdot a_{n-1})^{1/n} = G_{n-1}^{(n-1)/n},$$

thus $A_{n-1} \geq G_{n-1}$, which means that $Q(n-1)$ holds.

Geometric Mean vs Arithmetic Mean: 1/2

Now we can show that

Claim (⋆)

$$A_n \geq G_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. We know:

1. $A^{2^m} \geq G_{2^m}$ for all $m \in \mathbb{N}$,
2. if $A_k \geq G_k$ holds for some $k \in \mathbb{N}$, then also holds for $k-1$, i.e. $A_{k-1} \geq G_{k-1}$ is true.

Concluding, we can easily prove Claim (⋆). Fix $n \in \mathbb{N}$ and choose the smallest $m \in \mathbb{N}$ so that

$$2^{m-1} < n \leq 2^m.$$

By (1) we know $A_{2^m} \geq G_{2^m}$ holds.

Geometric Mean vs Arithmetic Mean: 2/2

Claim ($\star$)

$$A_n \geq G_n \quad \text{for all} \quad n \in \mathbb{N}.$$

By (2) with $k = 2^m$ we deduce

$$A_{2^m} \geq G_{2^m} \quad \text{implies} \quad A_{2^m-1} \geq G_{2^m-1}.$$

Repeating

$$A_{2^m-1} \geq G_{2^m-1} \quad \text{implies} \quad A_{2^m-2} \geq G_{2^m-2}.$$

We now apply (2) as many times until we reach $A_n \geq G_n$ and the proof is finally completed. □

Means

Let $a_1, a_2, \ldots, a_n > 0$ be given. We have the following means.

Arithmetic mean

$$A_n = \frac{a_1 + \ldots + a_n}{n};$$

Geometric mean

$$G_n = (a_1 \cdot \ldots \cdot a_n)^{1/n};$$

Harmonic mean

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}};$$

Quadratic mean

$$Q_n = \left(\frac{a_1^2 + \ldots + a_n^2}{n} \right)^{1/2}.$$

Theorem

Theorem

For all $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n > 0$ we have

$$\min(a_1, \ldots, a_n) \le H_n \le G_n \le A_n \le Q_n \le \max(a_1, \ldots, a_n).$$

Proof. We will proceed in several steps.

- We have proved that $A_n \ge G_n$.
- To prove $H_n \le G_n$ we apply $A_n \ge G_n$ with

$$\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}.$$

We obtain

$$G_n^{-1} = \left(\frac{1}{a_1} \cdot \frac{1}{a_2} \cdot \ldots \cdot \frac{1}{a_n}\right)^{1/n} \le \frac{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}}{n} = H_n^{-1},$$

thus $H_n \le G_n$.

Proof: 1/2

- To prove inequality $A_n \leq Q_n$ consider the relation

$$\begin{aligned}(a_1 + a_2 + \ldots + a_n)^2 = & \, a_1^2 + a_2^2 + \ldots + a_n^2 \\ & + 2(a_1 a_1 + a_1 a_3 + \ldots + a_1 a_n) \\ & + 2(a_2 a_3 + a_2 a_4 + \ldots + a_2 a_n) \\ & + \ldots + 2(a_{n-2} a_{n-1} + a_{n-2} a_n) + 2a_{n-1} a_n.\end{aligned}$$

Since $2a_i a_j \leq a_i^2 + a_j^2$, thus

$$(a_1 + \ldots + a_n)^2 \leq n(a_1^2 + \ldots + a_n^2).$$

Hence

$$a_1 + \ldots + a_n \leq \left(n(a_1^2 + \ldots + a_n^2)\right)^{1/2},$$

and consequently $A_n \leq Q_n$.

Proof: 2/2

- Finally wlog suppose that

$$0 < a_1 \le a_2 \le \ldots \le a_n.$$

then $a_1 = \min(a_1, \ldots, a_n)$, and $a_n = \max(a_1, \ldots, a_n)$. Hence

$$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} \ge n\frac{a_1}{n} = a_1,$$

and

$$Q_n = \left(\frac{a_1^2 + \ldots + a_n^2}{n}\right)^{1/2} \le \left(\frac{na_n^2}{n}\right)^{1/2} = a_n.$$

The proof is completed. □

AM-GM inequality - example

Example

Prove that for any $x, y, z > 0$ we have

$$\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy} \geq 3.$$

Solution. Consider the numbers $\frac{x^2}{yz}, \frac{y^2}{xz}, \frac{z^2}{xy}$. Then

$$A_3 = \frac{\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy}}{3},$$

$$G_3 = \sqrt[3]{\frac{x^2}{yz} \cdot \frac{y^2}{xz} \cdot \frac{z^2}{xy}} = 1,$$

so our inequality is a consequence of $A_3 \geq G_3$. □

AM-GM inequality - example

Example

If the product of n positive real numbers is 1, then their sum is at least n.

Solution. Let $a_1, \ldots, a_n > 0$ be the numbers such that

$$G_n = \sqrt[n]{a_1 \cdots a_n} = 1,$$

so by $A_n \geq G_n$,

$$a_1 + \ldots + a_n \geq nG_n = n.$$

□

Bernoulli inequality: 1/2

Bernoulli inequality

If $x > -1$ and $n \in \mathbb{N}$, then one has

$$(1+x)^n \geq 1 + nx.$$

Proof. We will use $A_n \geq G_n$ with

$$a_1 = a_2 = \ldots = a_{n-1} = 1 \quad \text{and} \quad a_n = 1 + nx.$$

Indeed,

$$A_n = \frac{a_1 + \ldots + a_n}{n} = \frac{\overbrace{1 + \ldots + 1}^{n-1 \text{ times}} + 1 + nx}{n} = \frac{n(1+x)}{n} = 1 + x.$$

Bernoulli inequality: 2/2

On the other hand

$$(1+x) = A_n \geq G_n = \Big(\overbrace{1 \cdot 1 \cdot \ldots \cdot 1}^{n-1 \text{ times}} \cdot (1+nx)\Big)^{1/n} = (1+nx)^{1/n},$$

which implies

$$(1+x)^n \geq 1+nx,$$

and we are done. □

Bernoulli inequality: generalization

Our aim will be to built tools and generalize Bernoulli's inequality. We show that the following is true.

Bernoulli inequality - generalization

If $-1 < x \neq 0$ and $a > 1$ or $a < 0$, then

$$(1+x)^a > 1 + ax.$$

If $-1 < x \neq 0$ and $0 < a < 1$, then

$$(1+x)^a < 1 + ax.$$

Cauchy–Schwarz inequality

Cauchy–Schwarz inequality

For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ one has

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right).$$

Proof. Consider the polynomial

$$0 \leq (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \ldots + (a_n x + b_n)^2 =$$
$$(a_1^2 + \ldots + a_n^2)x^2 + 2(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)x + (b_1^2 + \ldots + b_n^2).$$

Since the polynomial is nonnegative

$$\Delta = 4(a_1 b_1 + \ldots + a_n b_n)^2 - 4(a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2) \leq 0$$

and we are done. □

Minkowski's inequality: 1/3

Minkowski's inequality

For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ one has

$$\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}.$$

Proof. Let

$$S_n = \left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2}.$$

Minkowski's inequality: 2/3

Then

$$S_n^2 = \sum_{j=1}^{n} |a_j + b_j|^2 = \sum_{j=1}^{n} |a_j + b_j||a_j + b_j|$$
$$\le \sum_{j=1}^{n} |a_j + b_j||a_j| + \sum_{j=1}^{n} |a_j + b_j||b_j|.$$

By the Cauchy–Schwarz inequality,

$$S_n^2 \le$$
$$\underbrace{\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2}}_{=S_n} \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} + \underbrace{\left(\sum_{j=1}^{n} |a_j + b_j|^2\right)^{1/2}}_{=S_n} \left(\sum_{j=1}^{n} |b_j|^2\right)^{1/2}$$

Minkowski's inequality: 3/3

$$= S_n \left(\left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \right).$$

Thus we have proved **a bootstrap inequality**, i.e.

$$S_n^2 \leq S_n \left(\left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2} \right).$$

Hence

$$S_n \leq \left(\sum_{j=1}^{n} |a_j|^2 \right)^{1/2} + \left(\sum_{j=1}^{n} |b_j|^2 \right)^{1/2}.$$

□

Weighted arithmetic and geometric means

Theorem

For all positive real numbers $a_1, a_2, \ldots, a_n$ and all positive weights $q_1, q_2, \ldots, q_n$ satisfying the following convexity condition

$$q_1 + \ldots + q_n = 1,$$

we have

$$a_1^{q_1} \cdot \ldots \cdot a_n^{q_n} \leq q_1 a_1 + \ldots + q_n a_n.$$

If $q_1 = q_2 = \ldots = q_n = \frac{1}{n}$, then we have

$$a_1^{q_1} \cdot \ldots \cdot a_n^{q_n} = (a_1 \cdot \ldots \cdot a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n} = q_1 a_1 + \ldots + q_n a_n,$$

which recovers the inequality between geometric and arithmetic means.

Proof: 1/2

Proof: We first assume

$$q_1, \ldots, q_n \in \mathbb{Q} \quad \text{and} \quad q_1, \ldots, q_n > 0.$$

We can assume that $q_i = \frac{k_i}{m}$ for $1 \leq i \leq n$ and

$$k_1 + \ldots + k_n = m.$$

Invoking the inequality between geometric and arithmetic means we obtain

$$\begin{aligned}
\sum_{i=1}^{n} q_i a_i &= k_1 \frac{a_1}{m} + \ldots + k_n \frac{a_n}{m} \\
&\geq m \left(\left(\frac{a_1}{m} \right)^{k_1} \cdot \ldots \cdot \left(\frac{a_n}{m} \right)^{k_n} \right)^{1/m} \\
&= a_1^{k_1/m} \cdot \ldots \cdot a_n^{k_n/m} \\
&= a_1^{q_1} \cdot \ldots \cdot a_n^{q_n}.
\end{aligned}$$

Proof: 2/2

If now all weights $q_1, \dots, q_n > 0$ are real numbers, then for any $1 \leq i \leq n$, we choose a sequence of positive rationals $(q_{i,k})_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} q_{i,k} = q_i$$

and so that

$$\sum_{i=1}^{n} q_{i,k} = 1 \quad \text{for all} \quad k \in \mathbb{N}.$$

Then by the previous part

$$a_1^{q_{i,k}} \cdot \ldots \cdot a_n^{q_{n,k}} \leq q_{1,k} a_1 + \ldots + q_{n,k} a_n.$$

Passing with $k \to \infty$ we conclude that

$$a_1^{q_1} \cdot \ldots \cdot a_n^{q_n} \leq q_1 a_1 + \ldots + q_n a_n.$$

Here we used the following fact: if $a > 0$ and $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} a^{x_n} = a^{x_0}$. □

Hölder's inequality

Hölder's inequality

Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for any real numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ one has

$$\sum_{j=1}^{n} |x_j y_j| \le \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{1/p} \Big(\sum_{j=1}^{n} |y_j|^q \Big)^{1/q}.$$

Proof. By the previous theorem for any $a_1, b_1 > 0$ we have

$$a_1^{\frac{1}{p}} b_1^{\frac{1}{q}} \le \frac{1}{p} a_1 + \frac{1}{q} b_1,$$

which for $a_1 = a^p$ and $b_1 = b^q$ yields

(*)

$$ab \le \frac{1}{p} a^p + \frac{1}{q} b^q.$$

Proof of Hölder's inequality 1/2

Let

$$a_j := \frac{|x_j|}{\left(\sum_{j=1}^n |x_j|^p\right)^{1/p}}, \quad b_j := \frac{|y_j|}{\left(\sum_{j=1}^n |y_j|^q\right)^{1/q}}$$

Applying (*) to each $1 \leq j \leq n$ one gets

$$\begin{aligned}\sum_{j=1}^n a_j b_j &\leq \sum_{j=1}^n \left(\frac{1}{p} a_j^p + \frac{1}{q} b_j^q\right) \\ &= \sum_{j=1}^n \left(\frac{|x_j|^p}{p\left(\sum_{j=1}^n |x_j|^p\right)} + \frac{|y_j^q|}{q\left(\sum_{j=1}^n |y_j|^q\right)}\right) \\ &= \frac{1}{p}\frac{\sum_{j=1}^n |x_j|^p}{\sum_{j=1}^n |x_j|^p} + \frac{1}{q}\frac{\sum_{j=1}^n |y_j|^q}{\sum_{j=1}^n |y_j|^q} = \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

Proof of Hölder's inequality 2/2

Thus we have proved

$$\sum_{j=1}^{n} a_j b_j \leq 1,$$

which is equivalent to

$$\sum_{j=1}^{n} |x_j y_j| \leq \Big(\sum_{j=1}^{n} |x_j|^p \Big)^{1/p} \Big(\sum_{j=1}^{n} |y_j|^q \Big)^{1/q}$$

and the proof of Hölder's inequality is completed. □