Lecture 8

Stolz theorem and Euler's number
Upper and lower limits

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Stolz theorem

Stolz theorem

Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be two sequences so that

1. (i) $(y_n)_{n\in\mathbb{N}}$ strictly increases to $+\infty$, i.e. $y_n < y_{n+1}$ for all $n \in \mathbb{N}$ and
$$\lim_{n\to\infty} y_n = +\infty.$$
2. (ii) Also we have
$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a,$$

then
$$\lim_{n\to\infty} \frac{x_n}{y_n} = a.$$

Remark: It is a prototype of a l'Hôpital's rule.

Proof: 1/3

- Without loss of generality we may assume that $y_n > 0$, since $\lim_{n\to\infty} y_n = +\infty$ and thus we have $y_n > 0$ for large $n \in \mathbb{N}$.
- Since $\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = a$, there is $M > 0$ such that for $n \geq M$ we have
$$a - \frac{\varepsilon}{2} < \frac{x_n - x_{n-1}}{y_n - y_{n-1}} < a + \frac{\varepsilon}{2}.$$
So for $n \geq M$ we have
$$\left(a - \frac{\varepsilon}{2}\right)(y_n - y_{n-1}) < x_n - x_{n-1} < \left(a + \frac{\varepsilon}{2}\right)(y_n - y_{n-1}).$$
- Summing now from $k = M$ to $k = n$ for any $n \geq M$ we get
$$\left(a - \frac{\varepsilon}{2}\right)(y_n - y_{M-1}) = \sum_{k=M}^{n} \left(a - \frac{\varepsilon}{2}\right)(y_k - y_{k-1}) < \sum_{k=M}^{n} (x_k - x_{k-1})$$
$$= x_n - x_{M-1} < \sum_{k=M}^{n} \left(a + \frac{\varepsilon}{2}\right)(y_k - y_{k-1}) = \left(a + \frac{\varepsilon}{2}\right)(y_n - y_{M-1}).$$
since the above sums are telescoping.

Proof: 2/3

- Therefore dividing by $y_n - y_{M-1}$ for any $n \geq M$ we obtain
$$a - \frac{\varepsilon}{2} < \frac{x_n - x_{M-1}}{y_n - y_{M-1}} < a + \frac{\varepsilon}{2}.$$
So
$$\left| \frac{x_n - x_{M-1}}{y_n - y_{M-1}} - a \right| < \frac{\varepsilon}{2} \qquad \text{for } n \geq M.$$

- Observe that
$$\left| \frac{x_n}{y_n} - \left(1 - \frac{y_{M-1}}{y_n}\right) \frac{x_n - x_{M-1}}{y_n - y_{M-1}} \right| = \left| \frac{x_n}{y_n} - \frac{x_n - x_{M-1}}{y_n} \right| = \left| \frac{x_{M-1}}{y_n} \right|,$$
so the triangle inequality gives us for $n \geq M$ the inequalities
$$\begin{aligned} \left| \frac{x_n}{y_n} - a \right| &\leq \left| \frac{x_n}{y_n} - \left(1 - \frac{y_{M-1}}{y_n}\right) \frac{x_n - x_{M-1}}{y_n - y_{M-1}} \right| + \left| \left(1 - \frac{y_{M-1}}{y_n}\right) \frac{x_n - x_{M-1}}{y_n - y_{M-1}} - a \right| \\ &\leq \left| \frac{x_{M-1}}{y_n} \right| + \left| \left(1 - \frac{y_{M-1}}{y_n}\right) \right| \left| \frac{x_n - x_{M-1}}{y_n - y_{M-1}} - a \right| + |a| \frac{y_{M-1}}{y_n} \\ &< \frac{\varepsilon}{2} + \frac{|x_{M-1}| + |a| y_{M-1}}{y_n}. \end{aligned}$$

Proof: 3/3

So for $n \geq M$ we have

$$\left|\frac{x_n}{y_n} - a\right| < \frac{\varepsilon}{2} + \frac{|x_{M-1}| + |a| y_{M-1}}{y_n}.$$

Since $\lim_{n\to\infty} y_n = +\infty$, we may choose $N \geq M$ such that for $n \geq N$ we have

$$\frac{|x_{M-1}| + |a| y_{M-1}}{y_n} < \frac{\varepsilon}{2}.$$

Therefore for $n \geq N$ we get

$$\left|\frac{x_n}{y_n} - a\right| < \varepsilon$$

and the proof is finished. □

Example 1/2

Exercise

Let $k \in \mathbb{N}$ be fixed. Find the limit

$$\lim_{n\to\infty} \frac{1^k + \ldots + n^k}{n^{k+1}}.$$

Proof. We apply Stolz's theorem with

$$x_n = 1^k + \ldots + n^k, \qquad y_n = n^{k+1}.$$

Observe that $(y_n)_{n\in\mathbb{N}}$ is strictly increasing and $\lim_{n\to\infty} y_n = +\infty$. Therefore it suffices to compute

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{n^k}{n^{k+1} - (n-1)^{k+1}}.$$

Example 2/2

Using the binomial theorem we get

$$
\begin{aligned}
(n-1)^{k+1} &= \sum_{m=0}^{k+1} \binom{k+1}{m} n^m (-1)^{k+1-m} \\
&= n^{k+1} - (k+1)n^k + \sum_{m=0}^{k-1} \binom{k+1}{m} n^m (-1)^{k+1-m}
\end{aligned}
$$

So we have

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n^k}{n^{k+1} - (n-1)^{k+1}} &= \lim_{n\to\infty} \frac{n^k}{(k+1)n^k - \sum_{m=0}^{k-1} \binom{k+1}{m} n^m (-1)^{k+1-m}} \\
&= \lim_{n\to\infty} \frac{1}{(k+1) - \sum_{m=0}^{k-1} \binom{k+1}{m} n^{m-k} (-1)^{k+1-m}} \\
&= \frac{1}{(k+1)}.
\end{aligned}
$$

□

Application

Proposition

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences such that

1. $b_n > 0$, $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n = +\infty$,
2. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = g$.

then

$$\lim_{n\to\infty} \frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n} = g.$$

Proof. We apply Stolz's theorem with $x_n = a_1 + \ldots + a_n$ and $y_n = b_1 + \ldots + b_n$. Then the assumptions of the Stolz theorem are satisfied as $y_{n+1} - y_n = b_{n+1} > 0$ and $y_n \geq b_n$ both diverge to $+\infty$, and

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{n\to\infty} \frac{a_n}{b_n} = g.$$

Therefore $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = g$ and the proof is finished. □

Euler's sequences: 1/4

Consider two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ defined by

$$a_n = \left(1 + \frac{1}{n}\right)^n, \qquad b_n = \left(1 + \frac{1}{n}\right)^{n+1} \quad \text{for all} \quad n \in \mathbb{N}$$

We have the following properties.

1. Observe that $a_n < b_n$ for all $n \in \mathbb{N}$. Indeed,

$$a_n = \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n}\right)^{n+1} = b_n,$$

since $1 < 1 + \frac{1}{n}$ for all $n \in \mathbb{N}$.

Euler's sequences: 2/4

2. The sequence $(a_n)_{n\in\mathbb{N}}$ is strictly increasing, i.e.

$$a_n < a_{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. By the geometric-arithmetic mean inequality $G_{n+1} < A_{n+1}$ (which is strict unless $x_1 = x_2 = \ldots = x_{n+1}$) with

$$x_1 = 1 \quad \text{and} \quad x_2 = x_3 = \ldots = x_{n+1} = 1 + \frac{1}{n},$$

we obtain

$$G_{n+1} = \left(\left(1+\frac{1}{n}\right)^n\right)^{1/(n+1)} < \frac{1 + n\left(1+\frac{1}{n}\right)}{n+1} = 1 + \frac{1}{n+1} = A_{n+1}.$$

Thus

$$a_n = \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} = a_{n+1}.$$

□

Euler's sequences: 3/4

3. The sequence $(b_n)_{n\in\mathbb{N}}$ is strictly decreasing, i.e.

$$b_{n+1} < b_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. By the harmonic-geometric mean inequality $H_{n+1} < G_{n+1}$ (which is strict unless $x_1 = x_2 = \ldots = x_{n+1}$) with

$$x_1 = 1 \quad \text{and} \quad x_2 = x_3 = \ldots = x_{n+1} = 1 + \frac{1}{n-1} = \frac{n}{n-1}.$$

Then

$$H_{n+1} = \frac{n+1}{1 + n\frac{n-1}{n}} < \left(1 + \frac{1}{n-1}\right)^{n/(n+1)} = G_{n+1},$$

thus

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1} < \left(1 + \frac{1}{n-1}\right)^{n} = b_{n-1}. \quad \square$$

Euler's sequences: 4/4

Collecting (1),(2),(3) we have

$$2 = a_1 < a_n < a_{n+1} < b_{n+1} < b_n < b_1 = 4 \quad \text{for all} \quad n \geq 2.$$

Thus the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) a_n = \left(\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\right) \left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} a_n.$$

Euler number

The limit of the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ is called **the Euler number**

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1} = e \simeq 2,718\ldots.$$

Euler's number - fact

Fact

If $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, then

$$\lim_{n\to\infty} \left(1 + \frac{1}{a_n}\right)^{a_n} = e.$$

Proof. Let $\lim_{n\to\infty} a_n = +\infty$ and consider $b_n = \lfloor a_n \rfloor$. Then $b_n \leq a_n < b_n + 1$, hence

$$\left(1 + \frac{1}{b_n + 1}\right)^{b_n} < \left(1 + \frac{1}{a_n}\right)^{a_n} < \left(1 + \frac{1}{b_n}\right)^{b_n+1}.$$

Proof: 1/3

By the squeeze theorem it suffices to prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{b_n+1}\right)^{b_n}=\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n+1}=e$$

or even

$$\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n}=e.$$

- If $(b_n)_{n\in\mathbb{N}}$ were increasing then as a subsequence of $(n)_{n\in\mathbb{N}}$ we could conclude $\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n}=e$, since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$.
- But we only know that $\lim_{n\to\infty} b_n=+\infty$. It does not mean that $(b_n)_{n\in\mathbb{N}}$ is increasing.

Proof: 2/3

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ we can find $\widetilde{N}_\varepsilon \in \mathbb{N}$ so that $n \geq \widetilde{N}_\varepsilon$ implies

$$\left|\left(1+\frac{1}{n}\right)^n - e\right| < \varepsilon.$$

But $\lim_{n\to\infty} b_n = +\infty$ thus we can find $N_\varepsilon \in \mathbb{N}$ so that $n \geq N_\varepsilon$ implies $b_n \geq \widetilde{N}_\varepsilon$. In particular, we conclude that

$$\left|\left(1+\frac{1}{b_n}\right)^{b_n} - e\right| < \varepsilon$$

for all $n \geq N_\varepsilon$ and thus

$$\lim_{n\to\infty}\left(1+\frac{1}{b_n}\right)^{b_n} = e.$$

Consequently, $\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e$ as $\lim_{n\to\infty} a_n = +\infty$.

Proof: 3/3

Moreover,

$$\lim_{n\to\infty}\left(1-\frac{1}{a_n}\right)^{a_n}=e^{-1},$$

because

$$\lim_{n\to\infty}\left(1-\frac{1}{a_n}\right)^{a_n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{a_n-1}\right)^{a_n}}=\frac{1}{e}.$$

this implies

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n}=e \quad \text{if} \quad \lim_{n\to\infty}a_n=-\infty.$$

Example

Exercise

Find $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{4n}$.

Solution. Since $(2n)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}=e.$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{4n}&=\left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)\left(\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n}\right)\\&=e\cdot e=e^2.\end{aligned}$$

□

Example

Exercise

Find $\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{4n^2+1}$.

Solution. Since $(n^2+1)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1} = e.$$

Therefore,

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{4n^2+1}$$
$$= \left(\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{n^2+1}\right)^4 \left(\lim_{n\to\infty}\left(1+\frac{1}{n^2+1}\right)^{-3}\right) = e^4. \quad \square$$

Upper and lower limits

- We write that $s_n \xrightarrow[n\to\infty]{} +\infty$ if for every $M > 0$ there is $n \in \mathbb{N}$ such that $n \geq N$ implies $s_n \geq M$.
- Similarly, $s_n \xrightarrow[n\to\infty]{} -\infty$ if for every $M > 0$ there is an integer $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n \leq -M$.

Upper limit and lower limit

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

- **The upper limit** is defined by

$$\limsup_{n\to\infty} s_n = \inf_{k\geq 1} \sup_{n\geq k} s_n.$$

- **The lower limit** is defined by

$$\liminf_{n\to\infty} s_n = \sup_{k\geq 1} \inf_{n\geq k} s_n.$$

Proposition

Proposition

For a sequence $(s_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, the upper and lower limits always exist.

Proof. Let $\alpha_k = \sup_{n\geq k} s_n$. Then $\alpha_{k+1} \leq \alpha_k$ and

$$\limsup_{n\to\infty} s_n = \inf_{k\geq 1} \sup_{n\geq k} s_n \underbrace{=}_{(MCT)} \lim_{n\to\infty} \alpha_k \text{ (possible infinite!)}.$$

If $\beta_k = \inf_{n\geq k} s_n$, then $\beta_k \leq \beta_{k+1}$ and

$$\liminf_{n\to\infty} s_n = \sup_{k\geq 1} \inf_{n\geq k} s_n \underbrace{=}_{(MCT)} \lim_{n\to\infty} \beta_k \text{ (possible infinite!)} \quad \square.$$

Remark

We always have $\beta_k = \inf_{n\geq k} s_n \leq \sup_{n\geq k} s_n = \alpha_k$. Thus

$$\liminf_{n\to\infty} a_n = \lim_{k\to\infty} \beta_k \leq \lim_{k\to\infty} \alpha_k = \limsup_{n\to\infty} s_n.$$

Examples 1/3

Example 1

Consider $a_n = (-1)^n \frac{n+1}{n}$. Let

$$\beta_n = \sup \left\{ (-1)^n \frac{n+1}{n}, (-1)^{n+1} \frac{n+2}{n+1}, \ldots \right\},$$

then

$$\beta_n = \begin{cases} \frac{n+1}{n} & \text{if } n \text{ is even,} \\ \frac{n+2}{n+1} & \text{if } n \text{ is odd.} \end{cases}$$

Thus $\lim_{n\to\infty} \beta_n = 1$. Therefore

$$\limsup_{n\to\infty} a_n = 1.$$

Similarly

$$\liminf_{n\to\infty} a_n = -1.$$

Examples 2/3

Example 2

Let

$$a_n = \begin{cases} 0 \text{ if } n \text{ is odd,} \\ 1 \text{ if } n \text{ is even.} \end{cases}$$

Then

$$\beta_n = \sup\{a_m \ : \ m \geq n\} = 1,$$
$$\alpha_n = \inf\{a_m \ : \ m \geq n\} = 0.$$

Therefore

$$\limsup_{n\to\infty} a_n = 1,$$
$$\liminf_{n\to\infty} a_n = 0.$$

Examples 3/3

Example 3

Let $a_n = \frac{1}{n}$. Then

$$\beta_n = \sup\left\{\frac{1}{m} \ : \ m \geq n\right\} = \frac{1}{n},$$

so $\lim_{n\to\infty} \beta_n = 0$. Similarly

$$\alpha_n = \inf\left\{\frac{1}{m} \ : \ m \geq n\right\} = 0,$$

so $\lim_{n\to\infty} \alpha_n = 0$. Thus

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = 0.$$

Accumulation points of a sequence

Definition

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that $x \in \mathbb{R} \cup \{\pm\infty\}$ is an accumulation point of $(s_n)_{n\in\mathbb{N}}$ if

$$s_{n_k} \xrightarrow[k\to\infty]{} x$$

for some subsequence $(s_{n_k})_{k\in\mathbb{N}}$.

Theorem

Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Let E be the set of all accumulation points of $(s_n)_{n\in\mathbb{N}}$. Then

$$\limsup_{n\to\infty} s_n = s^* = \sup E,$$

$$\liminf_{n\to\infty} s_n = s_* = \inf E.$$

Proof: Case 1

Suppose that

$$\limsup_{n\to\infty} s_n = +\infty,$$

thus

$$\inf_{k\geq 1} \sup_{n\geq k} s_n = +\infty,$$

so

$$\sup_{n\geq k} s_n = +\infty \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence there is $(n_k)_{k\in\mathbb{N}}$ so that

$$\lim_{k\to\infty} s_{n_k} = +\infty.$$

this gives $s^* = \sup E = +\infty$. □

Proof: Case 2

Suppose that

$$\limsup_{n\to\infty} s_n = -\infty,$$

so

$$\lim_{k\to\infty} \sup_{n\geq k} s_n = -\infty.$$

This means that for every $M > 0$ there is $N \in \mathbb{N}$ so that $k \geq N$ implies

$$\sup_{n\geq k} s_n \leq -M.$$

Hence $s_n \leq -M$ for all $n \geq N$, i.e.

$$\lim_{n\to\infty} s_n = -\infty.$$

So $E = \{-\infty\}$ and $s^* = \sup E = -\infty$. □

Proof: Case 3

Claim:

Assume that $\limsup_{n\to\infty} s_n = L$ and $L \in \mathbb{R}$. Then

(a) $\sup E \le L$,

(b) $L \in E$,

which implies $s^* = \sup E = L$.

Remark:

This gives a stronger conclusion

$$\limsup_{n\to\infty} s_n = \sup E = \max E.$$

Proof: Case 3 proof of property (a): 1/2

- Suppose that $L < \sup E$. Thus there is $x \in E$ such that

$$L < x \leq \sup E,$$

and there exists a sequence $(s_{n_j})_{j\in\mathbb{N}}$ so that $\lim_{j\to\infty} s_{n_j} = x$, i.e. for every $\varepsilon > 0$ there exists $K_0 \in \mathbb{N}$ so that

$$j \geq K_0 \quad \text{implies} \quad |s_{n_j} - x| < \varepsilon.$$

- In particular, taking $\varepsilon = \frac{x-L}{2}$ we obtain that

$$\frac{x+L}{2} = x - \varepsilon < s_{n_j} \quad \text{for all} \quad j \geq K_0.$$

- Since $\limsup_{n\to\infty} s_n = \inf_{k\geq 1} \sup_{n\geq k} s_n = L$ we obtain that for every $\varepsilon > 0$ there is $K_\varepsilon \in \mathbb{N}$ so that

$$k \geq K_\varepsilon \quad \text{implies} \quad L \leq \sup_{n\geq k} s_n < L + \varepsilon.$$

Proof: Case 3 proof of property (a): 2/2

Taking $\varepsilon = \frac{x-L}{2}$ we obtain

$$\sup_{n \geq k} s_n < L + \varepsilon = \frac{x+L}{2}.$$

Thus picking $j_0 \geq K_0$ so that $n_{j_0} \geq K_\varepsilon$ we obtain

$$s_{n_{j_0}} \leq \sup_{n \geq K_\varepsilon} s_n < \frac{L+x}{2} < s_{n_{j_0}},$$

which is impossible. Thus (a) must be true, i.e.

$$\sup E \leq L.$$

Proof: Case 3 proof of property (b): 1/2

- We now construct $(s_{n_j})_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} s_{n_j} = L$.
- Since $\limsup_{n\to\infty} s_n = \inf_{k\geq 1}\sup_{n\geq k} s_n = L$ then for any $\varepsilon > 0$ there is $K_\varepsilon \in \mathbb{N}$ so that $k \geq K_\varepsilon$ implies

(*)

$$L \leq \sup_{n\geq k} s_n < L + \varepsilon.$$

- Let $\varepsilon = 1$ and let $K_1 \in \mathbb{N}$ so that (*) holds. Then there is $n_1 \in \mathbb{N}$ such that

$$L - 1 \leq \sup_{n\geq K_1} s_n - 1 < s_{n_1} < \sup_{n\geq K_1} s_n < L + 1.$$

- Suppose that we have constructed inductively a sequence $n_1 < n_2 < \ldots < n_j$ such that

$$L - \frac{1}{j} \leq s_{n_j} \leq L + \frac{1}{j}.$$

Proof: Case 3 proof of property (b): 2/2

- We now construct n_{j+1}. Set $\varepsilon = \frac{1}{j+1}$ in (*) which yields a corresponding $K_{1/(j+1)} \in \mathbb{N}$. Let $\widetilde{K}_j = \max(n_j, K_{1/(j+1)}) + 1$. Using (*) we see

$$L \leq \sup_{n \geq \widetilde{K}_j} s_n < L + \frac{1}{j+1}$$

and we find $n_{j+1} > \widetilde{K}_j > n_j$ such that

$$L - \frac{1}{j+1} \leq \sup_{n \geq \widetilde{K}_j} s_n - \frac{1}{j+1} < s_{n_{j+1}} \leq \sup_{n \geq \widetilde{K}_j} s_n < L + \frac{1}{j+1}$$

hence

$$\lim_{j \to \infty} s_{n_j} = L.$$

and we are done. □

Proposition

Proposition

A sequence $(s_n)_{n\in\mathbb{N}}$ is convergent and has a limit $L \in \mathbb{R} \cup \{\pm\infty\}$ iff

$$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n.$$

Proof. If $\lim_{n\to\infty} s_n = L$, then $E = \{L\}$ thus $s^* = s_* = L$ and by the previous theorem

$$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L.$$

Conversely, if $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$, then

$$\alpha_k = \inf_{n\geq k} s_n \leq s_k \leq \sup_{n\geq k} s_n = \beta_k$$

and $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = L$, thus $\lim_{n\to\infty} s_n = L$. □