Lecture 9

Infinite series and their properties

MATH 411H, FALL 2025

October 2, 2025

Series

Definition

We say that the series $\sum_{n=1}^{\infty} a_n$ **converges** to $A \in \mathbb{R}$ and write $\sum_{n=1}^{\infty} a_n = A$ if the associated sequence of its **partial sums**

$$s_n = \sum_{k=1}^{n} a_k = a_1 + \ldots + a_n \xrightarrow[n\to\infty]{} A.$$

If $(s_n)_{n\in\mathbb{N}}$ diverges the series $\sum_{n=1}^{\infty} a_n$ is said **to diverge**.

Remark

- Saying that the series $\sum_{n=1}^{\infty} a_n$ converges we understand that $|\sum_{k=1}^{\infty} a_k| < \infty$.
- Saying that the series $\sum_{n=1}^{\infty} a_n$ diverges we understand that $|\sum_{k=1}^{\infty} a_k| = \infty$.

Example

Exercise

If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$, the series diverges.

Solution. If $x < 1$, then

$$s_n = \sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

and the result follows if we let $n \to \infty$.

For $x \geq 1$ note that

$$\underbrace{1 + 1 + \ldots + 1}_{n} \leq s_n.$$

We have $\lim_{n\to\infty} n = +\infty$, thus $\lim_{n\to\infty} s_n = +\infty$.

Example

Exercise

$$\sum_{n=1}^{\infty} \frac{1}{k^2} < \infty.$$

Solution. Because the terms in the sum are all positive the sequence

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} \quad \text{is increasing.}$$

We now show that $(s_n)_{n\in\mathbb{N}}$ is bounded.

- The (MCT) will prove that the series converges.

Solution

To prove boundedness of $(s_n)_{n\in\mathbb{N}}$ we note that

$$\begin{aligned} s_n &= 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \ldots + \frac{1}{n\cdot n} \\ &< 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{(n-1)n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2. \end{aligned}$$

Thus the limit $\lim_{n\to\infty} s_n$ exists. □

- One can also prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. This is also Euler's result.

An example of a diverging series

Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Solution. Note that

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{16}\right) + \left(\frac{1}{17} + \ldots\right.$$

$$\geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16} + 16 \cdot \frac{1}{32} + \ldots$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = 1 + \lim_{n\to\infty} \frac{n}{2} = \infty.$$

Thus $s_n = \sum_{k=1}^{n} \frac{1}{k} \xrightarrow[n\to\infty]{} \infty$. □

Cauchy Condensation Test

Cauchy Condensation Test

Suppose that $(b_n)_{n\in\mathbb{N}}$ is decreasing and $b_n \geq 0$ for all $n \in \mathbb{N}$. Then the series

$$\sum_{n=1}^{\infty} b_n < \infty \quad \text{converges}$$

iff the series

$$\sum_{n=1}^{\infty} 2^n b_{2^n} < \infty \quad \text{converges.}$$

Proof. Let

$$s_n = b_1 + b_2 + \ldots + b_n,$$

$$t_k = b_1 + 2b_2 + \ldots + 2^k b_{2^k}.$$

Proof: 1/2

For $n < 2^k$ one has

$$\begin{aligned} s_n &\le b_1 + \overbrace{b_2 + b_3}^{2} + \ldots + \overbrace{b_{2^k} + \ldots + b_{2^{k+1}-1}}^{2^k} \\ &\le b_1 + 2b_2 + \ldots + 2^k b_{2^k} = t_k. \end{aligned}$$

(*)

so that $s_n \le t_k$ for $n < 2^k$.

Proof: 2/2

If $n > 2^k$ one has

$$s_n \geq b_1 + b_2 + (b_3 + b_4) + \ldots + (b_{2^{k-1}+1} + \ldots + b_{2^k})$$
$$\geq \frac{1}{2} b_1 + b_2 + 2b_4 + \ldots + 2^{k-1} b_{2^k} = \frac{1}{2} t_k.$$

(**)

Thus $2s_n \geq t_k$ for $n > 2^k$.

- By (*) and (**) the sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ are either both bounded or both unbounded. □

Corollary

Corollary

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \text{iff} \quad p > 1$$

Proof. The sequence $b_n = \frac{1}{n^p}$ is decreasing and $b_n \geq 0$ for all $n \in \mathbb{N}$. By the Cauchy condensation test we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \frac{2^n}{2^{pn}} < \infty.$$

But the latter converges provided that

$$\sum_{n=1}^{\infty} \frac{2^n}{2^{pn}} = \sum_{n=1}^{\infty} 2^{(1-p)n} = \frac{1}{1 - \frac{1}{2^{p-1}}} < \infty \quad \Longleftrightarrow \quad p > 1. \quad \square$$

Series representation of Euler's number

Theorem

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Proof. Let $s_n = \sum_{k=0}^{n} \frac{1}{k!}$. Then

1. $s_n < s_{n+1}$ for all $n \in \mathbb{N}$,
2. $s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} < 2 + \sum_{k=2} \frac{1}{2^{k-1}} < 3.$

Thus the limit $\lim_{n\to\infty} s_n$ exists.

Let $t_n = \left(1 + \frac{1}{n}\right)^n$, then $\lim_{n\to\infty} t_n = e$. By the binomial theorem

$$t_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}.$$

Proof: 1/2

Then

$$t_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$
$$= \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{1}{n^k}$$
$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \dots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \cdot \dots \cdot \left(1 - \frac{n-1}{n}\right) = \sum_{k=0}^{n} \frac{1}{k!} = s_n.$$

Thus

$$e = \lim_{n\to\infty} t_n \leq \lim_{n\to\infty} s_n.$$

Proof: 2/2

Next if $n \geq m$

$$t_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).$$

Let $n \to \infty$ keeping m fixed, we get

$$e = \lim_{n\to\infty} t_n \geq \sum_{k=0}^{m} \frac{1}{k!}.$$

Letting $m \to \infty$ we see $\lim_{m\to\infty} s_m \leq e$.

$$\lim_{m\to\infty} s_m = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{1}{k!} = e.$$

This completes the proof of the theorem. $\square$

Remark

We have $s_n = \sum_{k=0}^{n} \frac{1}{k!} < e$ for all $n \in \mathbb{N}$. Indeed

$$\begin{aligned} e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots \\ &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots\right) \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) \\ &\leq \frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+1}} = \frac{1}{(n+1)!}\frac{n+1}{n} = \frac{1}{n!n}. \end{aligned}$$

Hence we conclude

The error estimate (*)

$$0 < e - s_n < \frac{1}{n!n}.$$

Euler's number e is irrational

Theorem

The Euler number e is irrational.

Proof. Suppose e is rational. Then $e = \frac{p}{q}$ where $p, q \in \mathbb{N}$. By (*) we have

$$0 < q!(e - s_q) < \frac{1}{q}.$$

By our assumption

$$q!e \in \mathbb{N} \quad \text{is an integer.}$$

Since

$$q!s_q = q!\left(1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{q!}\right) \in \mathbb{N},$$

we see $q!(e - s_q) \in \mathbb{N}$, but if $q > 1$ and this is impossible since

$$0 < q!(e - s_q) < 1/q < 1.$$

Hence e must be irrational. □

Algebraic limit theorem for series

Algebraic limit theorem for series

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ then

$$\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) = \alpha A + \beta B.$$

Proof. Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$. We know that

$$\lim_{n\to\infty} A_n = A, \quad \text{and} \quad \lim_{n\to\infty} B_n = B,$$

so

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n}(\alpha a_k + \beta b_k) &= \lim_{n\to\infty} \alpha \sum_{k=1}^{n} a_k + \beta \sum_{k=1}^{n} b_k \\ &= \alpha \lim_{n\to\infty} A_n + \beta \lim_{n\to\infty} B_n = \alpha A + \beta B.\end{aligned}$$

□

Cauchy Criterion for Series

Theorem

The series $\sum_{k=1}^{\infty} a_k$ converges iff for every $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that whenever $n > m \geq N_\varepsilon$ it follows

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ and we show that $(s_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Observe that whenever $n > m \geq N_\varepsilon$ then

$$|s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

We now apply the **Cauchy Criterion for sequences** and we are done. □

Theorem

Theorem

If the series $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $\varepsilon > 0$ be given. Apply the previous theorem with $m = n-1$, then

$$|a_n| = |s_n - s_{n-1}| < \varepsilon$$

whenever $n > N_\varepsilon$, and we are done. □

Remark

But $\lim_{n\to\infty} a_n = 0$ does not imply $|\sum_{k=1}^{\infty} a_k| < \infty$.

- Consider $a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$, but $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Example

Exercise

Determine if the series

$$\sum_{n=1}^{\infty}(-1)^n\left(1-\frac{1}{n^3}\right)^{n^2}$$

diverges or converges.

Solution. Since $(n^3)_{n\in\mathbb{N}}$ is a subsequence of $(n)_{n\in\mathbb{N}}$ we have

$$\lim_{n\to\infty}\left(1-\frac{1}{n^3}\right)^{n^3}=e^{-1},$$

hence $\lim_{n\to\infty}\left(1-\frac{1}{n^3}\right)^{n^2}=1$, and the limit $\lim_{n\to\infty}(-1)^n\left(1-\frac{1}{n^3}\right)^{n^2}$ does not exist, so the series **diverges**. □

Comparison test

Comparison test

Assume that sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ satisfy

$$0 \le a_k \le b_k \quad \text{for all} \quad k \in \mathbb{N}.$$

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof. Both statements follows from the Cauchy Criterion for series:

$$\left|\sum_{k=m+1}^{n} a_k\right| \le \left|\sum_{k=m+1}^{n} b_k\right|.$$

This completes the proof. □

Example

Exercise

Determine if the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n} + 15}$$

diverges or converges.

Solution. For all $n \in \mathbb{N}$ we have

$$\frac{1}{n^2 + \sqrt{n} + 15} \leq \frac{1}{n^2}, \qquad \text{thus}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + \sqrt{n} + 15} < \infty. \qquad \square$$

Example

Exercise

Determine if the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} + \sqrt{n} + 1}$$

diverges or converges.

Solution. For all $n \in \mathbb{N}$ we have

$$\frac{1}{\sqrt[3]{n} + \sqrt{n} + 1} \geq \frac{1}{3\sqrt{n}}, \qquad \text{thus}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

hence

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} + \sqrt{n} + 1} = \infty. \qquad \square$$

Theorem

Theorem

A series of nonnegative terms $a_k \geq 0$ converges iff its partial sums form a bounded sequence.

Proof. If $\sum_{k=1}^{\infty} a_k < \infty$ one sees that

$$s_N = \sum_{k=1}^{N} a_k \leq M = \sum_{k=1}^{\infty} a_k < \infty.$$

Conversely, we also know that $s_N \leq s_{N+1} \leq M$ for all $N \in \mathbb{N}$. Then the limit

$$\lim_{N \to \infty} s_N$$

exists by the (MCT). $\square$

Root test

Root test

Given $\sum_{n=1}^{\infty} a_n$ set

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

(a) If $\alpha < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\alpha > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(c) If $\alpha = 1$, no information.

Proof. If $\alpha < 1$ we can choose β so that $\alpha < \beta < 1$ and the integer $N \in \mathbb{N}$ so that

$$\sqrt[n]{|a_n|} < \beta \quad \text{for all} \quad n \geq N,$$

since

$$\alpha = \inf_{k\geq 1} \sup_{n\geq k} \sqrt[n]{|a_n|} < \beta.$$

Proof

- For $n \geq N$ we have $|a_n| < \beta^n$, but $\beta < 1$, thus $\sum_{n=1}^{\infty} \beta^n$ converges and the comparision test implies that $\sum_{n=1}^{\infty} a_n$ converges as well.
- If $\alpha > 1$ then there is $(n_k)_{k\in\mathbb{N}}$ so that

$$|a_{n_k}|^{1/n_k} \xrightarrow[k\to\infty]{} \alpha.$$

  Hence $|a_n| > 1$ holds for infinitely many values of $n \in \mathbb{N}$, so that the condition $a_n \xrightarrow[n\to\infty]{} 0$ necessary for convergence $\sum_{n=1}^{\infty} a_n$ does not hold.
- To prove (c) note that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{and} \quad \sqrt[n]{n} \xrightarrow[n\to\infty]{} 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad \text{and} \quad \sqrt[n]{n^2} \xrightarrow[n\to\infty]{} 1.$$

  This completes the proof. □

Examples

Example 1

$$\sum_{n=1}^{\infty} \left(\frac{e}{n}\right)^n < \infty,$$

since

$$\sqrt[n]{\frac{e^n}{n^n}} = \frac{e}{n} \xrightarrow[n\to\infty]{} 0.$$

Example 2

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} < \infty,$$

since

$$\sqrt[n]{\frac{n^2}{2^n}} \xrightarrow[n\to\infty]{} \frac{1}{2}.$$

Ratio test

Ratio test

The series $\sum_{n=1}^{\infty} a_n$

- (a) converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$,
- (b) diverges if $\left|\frac{a_{n+1}}{a_n}\right| > 1$ for all $n \geq n_0$ for some fixed $n_0 \in \mathbb{N}$.

Proof. If (a) holds we can find $\beta < 1$ and $n \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \quad \text{for all} \quad n \geq N.$$

Proof

- In particular, for $p \in \mathbb{N}$, one has

$$\begin{aligned} |a_{n+p}| &= |a_{n+p-1}| \frac{|a_{n+p}|}{|a_{n+p-1}|} \\ &< \beta |a_{n+p-1}| \\ &< \beta^2 |a_{n+p-2}| < \ldots < \\ &< \beta^p |a_n|. \end{aligned}$$

- Thus $|a_{N+p}| < \beta^p |a_N|$ and

$$|a_n| < |a_N| \beta^{-N} \beta^n \quad \text{for all} \quad n \geq N.$$

- The claim follows from the comparison test since $\sum_{n=1}^{\infty} \beta^n < \infty$ whenever $\beta < 1$.
- If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$ then $a_n \xrightarrow[n\to\infty]{} 0$ does not hold. □

Remark and example

Remark

As before $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ is useless:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \text{and} \quad \frac{a_{n+1}}{a_n} = \frac{n}{n+1} \xrightarrow[n\to\infty]{} 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \quad \text{and} \quad \frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^2 \xrightarrow[n\to\infty]{} 1.$$

Example

$\sum_{n=1}^{\infty} \frac{n!}{n^n} < \infty$, since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e} < 1.$$