Lecture 10

Carathéodory's extension theorem and applications

Graduate Real Analysis

Fall Semester

Premeasures

Definition of a premeasure

A **premeasure** on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ of sets on X is a set function $\mu_0 : \mathcal{A} \to [0, \infty]$ satisfying

1. $\mu_0(\emptyset) = 0$,
2. if $(A_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$, then
$$\mu_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu_0(A_n).$$

Remark

- In other words, a premeasure $\mu_0 : \mathcal{A} \to [0, \infty]$ on an algebra $\mathcal{A}$ is a countably additive set function such that $\mu_0(\emptyset) = 0$.
- A premeasure is always finitely additive.
- The notions of finite and σ-finite premeasures are defined just as for measures.

From countably subadditive set functions to premeasures

Theorem

Let $\rho : \mathcal{E} \to [0, \infty]$ be a set function on a semi-algebra $\mathcal{E}$ of subsets of a set X and $\rho(\emptyset) = 0$. Suppose that ρ is finitely additive on $\mathcal{E}$ and let $\mathcal{A}$ be the algebra that consists of all finite disjoint unions of members of $\mathcal{E}$. Define a set function μ_0 on $\mathcal{A}$ by setting

$$\mu_0(A) = \sum_{i=1}^{n} \rho(E_i)$$

for $A = \bigcup_{j=1}^{n} E_j \in \mathcal{A}$, where $E_1, \ldots, E_n \in \mathcal{E}$ and $E_i \cap E_j = \emptyset$ if $i \neq j$.

a. Then μ_0 is a well-defined additive set function on $\mathcal{A}$ such that $\mu_0(\emptyset) = 0$ and $\mu_0 = \rho$ on $\mathcal{E}$.
b. If additionally ρ is countably subadditive on $\mathcal{E}$ then μ_0 is a premeasure on $\mathcal{A}$.

Proof of (a): 1/2

Proof of (a): To prove that μ_0 is well defined it suffices to show that for any finite collection of disjoint sets $(E_i)_{i=1}^m \subseteq \mathcal{E}$ and $(F_j)_{j=1}^n \subseteq \mathcal{E}$ such that

$$\bigcup_{i=1}^m E_i = \bigcup_{j=1}^n F_j,$$

one has

$$\sum_{i=1}^m \rho(E_i) = \sum_{j=1}^n \rho(F_j).$$

Let $G_{i,j} = E_i \cap F_j$, then $\{G_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$ is a collection of disjoint sets in $\mathcal{E}$ and

$$E_i = \bigcup_{j=1}^n G_{i,j}, \quad F_j = \bigcup_{i=1}^m G_{i,j}.$$

Proof of (a): 2/2

Since ρ is finitely additive on $\mathcal{E}$ we obtain

$$\rho(E_i) = \mu_0(E_i) = \sum_{j=1}^{n} \rho(G_{i,j}),$$

and

$$\rho(F_j) = \mu_0(F_j) = \sum_{i=1}^{m} \rho(G_{i,j}).$$

Consequently

$$\sum_{i=1}^{m} \rho(E_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(G_{i,j}) = \sum_{j=1}^{n} \sum_{i=1}^{m} \rho(G_{i,j}) = \sum_{j=1}^{n} \rho(F_j).$$

Thus μ_0 is well defined finitely additive set function on $\mathcal{A}$ such that $\mu_0(E) = \rho(E)$ for every $E \in \mathcal{E}$. $\square$

Proof of (b): 1/5

Proof of (b): To prove the second part assume that ρ is countably subadditive on $\mathcal{E}$. We first prove that ρ is countably additive on $\mathcal{E}$. Let $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{E}$ be a collection of disjoint sets such that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{E}$. Consider μ_0 corresponding to ρ as in part (a). For every $N \in \mathbb{N}$ we have

$$\bigcup_{n=1}^{N} E_n \subseteq \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{E} \subseteq \mathcal{A}.$$

Since $\bigcup_{n=1}^{N} E_n \in \mathcal{A}$, we conclude

$$\rho\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \mu_0\left(\bigcup_{n\in\mathbb{N}} E_n\right) \geq \mu_0\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \mu_0(E_n) = \sum_{n=1}^{N} \rho(E_n).$$

Proof of (b): 2/5

Passing with $N \to \infty$ we have

$$\rho\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) \geq \sum_{n=1}^{\infty} \rho(E_n).$$

Since ρ is also countably subadditivity we have

$$\rho\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) \leq \sum_{n=1}^{\infty} \rho(E_n).$$

Thus ρ is countably additive on $\mathcal{E}$.

- We now have to show that countable additivity of ρ on $\mathcal{E}$ ensures countable additivity of μ_0 on $\mathcal{A}$. In other words, we have to prove that μ_0 is a premeasure on $\mathcal{A}$.

Proof of (b): 3/5

Let $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be a sequence of disjoint sets so that $A = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$. By the definition of $\mathcal{A}$ for each $n \in \mathbb{N}$ there is a finite disjoint collection $(F_{n,j_n})_{j_n=1}^{k_n} \subseteq \mathcal{E}$ for some $k_n \in \mathbb{N}$ such that

$$A_n = \bigcup_{j_n=1}^{k_n} F_{n,j_n}.$$

Let $\mathcal{F} = \{F_{n,j_n} : n \in \mathbb{N},\ 1 \leq j_n \leq k_n\} \subseteq \mathcal{E}$. Since $A \in \mathcal{A}$ thus there exists a finite disjoint collection $\mathcal{G} = (E_i)_{i=1}^m \subseteq \mathcal{E}$ so that

$$A = \bigcup_{i=1}^{m} E_i.$$

We see that $F_{n,j_n} \cap E_i \in \mathcal{E}$.

Proof of (b): 4/5

If we let

$$\mathcal{H} = \{F_{n,j_n} \cap E_i : n \in \mathbb{N},\ 1 \leq j_n \leq k_n,\ 1 \leq i \leq m\}$$

then by the disjointness of the collection $\mathcal{F}$ and the disjointness of the collection $\mathcal{G}$ we see that $\mathcal{H}$ is a disjoint collection in $\mathcal{E}$ as well.

$$A_n \cap E_i = \bigcup_{j=1}^{k_n} F_{n,j_n} \cap E_i,$$

$$E_i = A \cap E_i = \bigcup_{n \in \mathbb{N}} \bigcup_{j_n=1}^{k_n} F_{n,j_n} \cap E_i.$$

Since $A_n \in \mathcal{A}$, and $E_i \in \mathcal{E} \subseteq \mathcal{A}$ we deduce that $A_n \cap E_i \in \mathcal{A}$.

Proof of (b): 5/5

By the definition of μ_0 and disjointness of collection $\mathcal{H}$ we obtain

$$\mu_0(A_n \cap E_i) = \sum_{j_n=1}^{k_n} \rho(F_{n,j_n} \cap E_i),$$

and by countable additivity of ρ we have

$$\rho(E_i) = \sum_{n\in\mathbb{N}} \sum_{j_n=1}^{k_n} \rho(F_{n,j_n} \cap E_v) = \sum_{n\in\mathbb{N}} \mu_0(A_n \cap E_i)$$

hence

$$\mu_0(A) = \sum_{i=1}^{m} \rho(E_i) = \sum_{n\in\mathbb{N}} \sum_{i=1}^{m} \mu_0(A_n \cap E_i) = \sum_{n\in\mathbb{N}} \mu_0(A_n).$$

Hence μ_0 is countably additive on $\mathcal{A}$ as desired. $\square$

Proposition

Proposition

Let $X \neq \emptyset$ be a set and let μ_0 be a premeasure on an algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ then an outer measure μ^* **induced** by μ_0 is defined by

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) : (E_j)_{j \in \mathbb{N}} \subseteq \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}, \qquad A \subseteq \mathcal{P}(X).$$

Then one has that

(a) $\mu_0(E) = \mu^*(E)$ for all $E \in \mathcal{A}$,

(b) $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$. In other words, every set in $\mathcal{A}$ is μ^*-measurable.

Proof of (a). Suppose $E \in \mathcal{A}$, we will show that $\mu_0(E) = \mu^*(E)$.

- Taking $E_1 = E$ and $E_2 = E_3 = \ldots = \emptyset$ we see that $E \subseteq \bigcup_{j=1}^{\infty} E_j$ and consequently $\mu^*(E) \leq \mu_0(E)$.

Proof: 1/3

- We now reverse this inequality and show that $\mu_0(E) \leq \mu^*(E)$. If $E \subseteq \bigcup_{j=1}^{\infty} A_j$ with $A_j \in \mathcal{A}$, then we let

$$B_n = E \cap \left(A_n \setminus \bigcup_{j=1}^{n-1} A_j \right) \in \mathcal{A}.$$

Then $B_n \cap B_m = \emptyset$ if $n \neq m$ and $E \subseteq \bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} A_j$. Now

$$\mu_0(E) \leq \sum_{j=1}^{\infty} \mu_0(B_j) \leq \sum_{j=1}^{\infty} \mu_0(A_j).$$

It follows that $\mu_0(E) \leq \mu^*(E)$ as claimed.

Proof: 2/3

Proof of (b). We now prove that $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$.

- Let $C \in \mathcal{A}$ and $E \subseteq X$ it suffices to show that

$$\mu^*(E) \geq \mu^*(E \cap C) + \mu^*(E \cap C^c).$$

Let $\varepsilon > 0$, then there is a sequence $(B_n)_{n=1}^{\infty} \subseteq \mathcal{A}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} B_n \quad \text{and} \quad \sum_{n=1}^{\infty} \mu_0(B_n) \leq \mu^*(E) + \varepsilon.$$

μ_0 is additive on $\mathcal{A}$, thus

$$\mu^*(E) + \varepsilon \geq \sum_{j=1}^{\infty} \mu_0(B_j) = \sum_{j=1}^{\infty} (\mu_0(B_j \cap C) + \mu_0(B_j \cap C^c)).$$

Proof: 3/3

- Since $\mu_0 \geq \mu^*$ and $B = \bigcup_{j=1}^{\infty} B_j$ we obtain

$$\begin{aligned}
\mu^*(E) + \varepsilon &\geq \sum_{j=1}^{\infty} (\mu_0(B_j \cap C) + \mu_0(B_j \cap C^c)) \\
&\geq \sum_{j=1}^{\infty} (\mu^*(B_j \cap C) + \mu^*(B_j \cap C^c)) \\
&\geq (\mu^*(B \cap C) + \mu^*(B \cap C^c)) \\
&\geq (\mu^*(E \cap C) + \mu^*(E \cap C^c))
\end{aligned}$$

by subadditivity of μ^*. Letting $\varepsilon \to 0$ we conclude

$$\mu^*(E) \geq \mu^*(E \cap C) + \mu^*(E \cap C^c),$$

thus C is μ^*-measurable and $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$ as desired. $\square$

Carathéodory's Theorem

Carathéodory's Theorem

Let μ^* be an outer measure on X, and let

$$\mathcal{M}(\mu^*) = \{A \subseteq X : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all} \quad E \subseteq X\}.$$

be the collection of μ^*-measurable sets. Then

1. $\mathcal{M}(\mu^*)$ is a σ-algebra, (**Carathéodory's σ-algebra**).
2. $(X, \mathcal{M}(\mu^*), \mu^*)$ is a complete measure space.

Carathéodory's extension theorem

Theorem

Let $X \neq \emptyset$ be a set, let $\mathcal{A} \subseteq \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$, and $\mathcal{M} = \sigma(\mathcal{A})$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0 and $\mu(E) = \mu^*(E)$ for all $E \in \mathcal{M}$, where μ^* is given by

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \mu_0(E_j) : (E_j)_{j \in \mathbb{N}} \subseteq \mathcal{A} \text{ and } A \subseteq \bigcup_{j=1}^{\infty} E_j \right\}, \qquad A \subseteq \mathcal{P}(X).$$

1. If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$.
2. If μ_0 is σ-finite, then μ is the unique extension of μ_0 to a measure on $\mathcal{M}$.

Proof: 1/3

Proof. For the first part of the theorem note that:

- $(X, \mathcal{M}(\mu^*), \mu^*)$ is a complete measure space, which follows from the Carathéodory theorem.
- $\mathcal{A} \subseteq \mathcal{M}(\mu^*)$ by the previous proposition and consequently we obtain that $\mathcal{M} = \sigma(\mathcal{A}) \subseteq \mathcal{M}(\mu^*)$.
- Thus it suffices to define μ to be the restriction of μ^* to $\mathcal{M} = \sigma(\mathcal{A})$. Then we see that $\mu(E) = \mu_0(E)$ for all $E \in \mathcal{A}$.

Proof of (i). Assume that ν is another measure on $\mathcal{M} = \sigma(\mathcal{A})$ such that $\nu(E) = \mu_0(E)$ for all $E \in \mathcal{A}$. We show that

(a) $\nu(E) \leq \mu(E)$ for all $E \in \sigma(\mathcal{A})$;

(b) $\nu(E) = \mu(E)$ if $E \in \sigma(\mathcal{A})$ and $\mu(E) < \infty$.

Proof: 2/3

For (a): If $E \in \mathcal{M}$ and $E \subseteq \bigcup_{j\in\mathbb{N}} A_j$, where $(A_j)_{j\in\mathbb{N}} \in \mathcal{A}$ then

$$\nu(E) \leq \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j)$$

and we conclude that $\nu(E) \leq \mu(E)$.
For (b): If we set $A = \bigcup_{j=1}^{\infty} A_j$, then we have

$$\nu(A) = \lim_{n\to\infty} \nu\Big(\bigcup_{j=1}^{n} A_j\Big) = \lim_{n\to\infty} \mu_0\Big(\bigcup_{j=1}^{n} A_j\Big) = \lim_{n\to\infty} \mu\Big(\bigcup_{j=1}^{n} A_j\Big) = \mu(A).$$

If $\mu(E) < \infty$, let $\varepsilon > 0$ and choose A_j's from $\mathcal{A}$ such that

$$\mu(A) \leq \mu(E) + \varepsilon,$$

hence $\mu(A \setminus E) < \varepsilon$ and

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) \leq \nu(E) + \varepsilon.$$

Proof 3/3

Letting $\varepsilon \to 0$ we see $\mu(E) \le \nu(E)$ if $E \in \mathcal{M} = \sigma(\mathcal{A})$ and $\mu(E) < \infty$ and we conclude in this case that

$$\nu(E) = \mu(E).$$

Proof of (ii). Finally suppose that $X = \bigcup_{j=1}^{\infty} A_j$, with $\mu_0(A_j) < \infty$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Then for any $E \in \mathcal{M}$ we have

$$\mu(E) = \sum_{j=1}^{\infty} \mu(E \cap A_j) = \sum_{j=1}^{\infty} \nu(E \cap A_j) = \nu(E).$$

This competes the proof of Carathéodory's extension theorem. □

Lebesgue–Stieltjes measure revised

Definition

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function. The **Lebesgue–Stieltjes outer measure** is defined for any $E \subseteq \mathbb{R}$ by

$$\mu_F^*(E) = \inf \left\{ \sum_{n\in\mathbb{N}} (F(b_n) - F(a_n)) : E \subseteq \bigcup_{n\in\mathbb{N}} (a_n, b_n] \right\}.$$

- The **Lebesgue–Stieltjes measure** μ_F is the restriction of μ_F^* to the Carathéodory σ-algebra $\mathcal{L}_{\mu_F} = \mathcal{M}(\mu_F^*)$ of μ_F^*-measurable sets.
- $\mathcal{L}_{\mu_F} = \mathcal{M}(\mu_F^*)$ will be called the **Lebesgue–Stieltjes σ-algebra**.
- The members of $\mathcal{L}_{\mu_F}$ are the **Lebesgue–Stieltjes measurable sets**.
- By Carathéodory's theorem $(\mathbb{R}, \mathcal{L}_{\mu_F}, \mu_F)$ is a complete measure space.
- Moreover, $(\mathbb{R}, \mathcal{L}_{\mu_F}, \mu_F)$ is a σ-finite measure space and $\mathrm{Bor}(\mathbb{R}) \subseteq \mathcal{L}_{\mu_F}$ and $\mu_F((a,b]) = F(b) - F(a)$ for any $-\infty \leq a \leq b \leq \infty$.

Set functions on a semi-algebra J_{oc}

Set functions on J_{oc} induced by increasing and right-continuous functions

Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right-continuous function. We define a set function $\rho_F : J_{oc} \to [0, \infty]$ by setting $\rho_F(\emptyset) = 0$ and

$$\rho_F(I) = F(b) - F(a) \quad \text{whenever} \quad I = (a, b] \in J_{oc},$$

with the understanding that $(a, \infty] = (a, \infty)$ and $F(\infty) = \lim_{x\to\infty} F(x)$.

Remark

(a) Let $\mathcal{A}$ be the algebra of finite disjoint unions of elements from J_{oc}. Then it is clear that $\sigma(\mathcal{A}) = \sigma(J_{oc}) = \mathrm{Bor}(\mathbb{R})$.

(b) It is easy to see that ρ_F is finitely additive on the semi-algebra J_{oc}, i.e. for any disjoint family $(I_j)_{j=1}^n \subseteq J_{oc}$ such that $\bigcup_{j=1}^n I_j \in J_{oc}$ we have

$$\rho_F\left(\bigcup_{j=1}^n I_j\right) = \sum_{j=1}^n \rho_F(I_j).$$

Our goal

A finitely additive set function on $\mathcal{A}$

Define a set function $\mu_{F,0}$ on $\mathcal{A}$ by setting

$$\mu_{F,0}(E) = \sum_{i=1}^{n} \rho_F(I_i)$$

for $E = \bigcup_{j=1}^{n} I_j \in \mathcal{A}$, where $(I_j)_{j=1}^{n} \subseteq J_{oc}$ and $I_i \cap I_j = \emptyset$ if $i \neq j$.

- By the first theorem and last remark $\mu_{F,0}$ is well defined and finitely additive on $\mathcal{A}$.
- Our goal is to show that ρ_F is countably subadditive on J_{oc}. Then by the first theorem we conclude that ρ_F is countably additive on J_{oc}, which in turn implies that its extension $\mu_{F,0}$ to $\mathcal{A}$ is also countably additive on $\mathcal{A}$. In other words, $\mu_{F,0}$ is a premeasure on $\mathcal{A}$.

ρ_F is countably subadditive on J_{oc}

Theorem

The set function $\rho_F : J_{oc} \to [0, \infty]$ defined above is countably subadditive on J_{oc}, which ensures that its extension $\mu_{F,0}$ to $\mathcal{A}$ is a premeasure on $\mathcal{A}$.

Proof. We have to prove that for any $(I_n)_{n\in\mathbb{N}} \subseteq J_{oc}$ such that $I = \bigcup_{n\in\mathbb{N}} I_n \in J_{oc}$ we have

$$\rho_F(I) \leq \sum_{n\in\mathbb{N}} \rho_F(I_n).$$

We first assume that $\rho_F(I) < \infty$.

- If $I = \emptyset$ then of course $\rho_F(I) = 0 \leq \sum_{n\in\mathbb{N}} \rho_F(I_n)$.
- If $\rho_F(I_n) = \infty$ for some $n \in \mathbb{N}$, then there is nothing to prove, since

$$\rho_F(I) < \infty = \sum_{n\in\mathbb{N}} \rho_F(I_n).$$

Proof

- We may assume that $\rho_F(I_n) < \infty$ for all $n \in \mathbb{N}$.
- Let $I = (a, b]$ for some $a < b$ and let $H_x = (x, \infty)$ and define

$$A = \Big\{ x \in [a, b] : F(b) - F(x) \le \sum_{j \in \mathbb{N}} \rho_F(I_j \cap H_x) \Big\}.$$

- Note that if $I_j = (c_j, d_j]$ then $I_j \cap H_x = (\max\{c_j, x\}, d_j]$, so the quantity $\rho_F(I_j \cap H_x) = F(d_j) - F(\max\{c_j, x\})$ is always defined.
- We have that $b \in A$ since

$$F(b) - F(b) = 0 \le \sum_{j \in \mathbb{N}} \rho_F(I_j \cap H_x) \quad \text{and} \quad A \subseteq [a, b].$$

- Thus $c = \inf A$ is defined and $c = \inf A \in [a, b]$. We show that $c \in A$. Indeed, right-continuity of F implies

$$F(b) - F(c) = \inf_{x \in A} (F(b) - F(x)) \le \inf_{x \in A} \sum_{j \in \mathbb{N}} \rho_F(I_j \cap H_x) \le \sum_{j \in \mathbb{N}} \rho_F(I_j \cap H_c),$$

since $\rho_F(I_j \cap H_x) \le \rho_F(I_j \cap H_c)$ as $c \le x \in A$.

Proof

- We now prove that $c = a$. If $a < c$, then $c \in (a, b]$ so there is $k \in \mathbb{N}$ such that $c \in I_k = (c_k, d_k]$, then $x = \max\{c_k, a\} < c \leq b$.
- For each $j \in \mathbb{N}$ we have

$$\rho_F(I_j \cap H_c) \leq \rho_F(I_j \cap H_x)$$

while

$$\rho_F(I_k \cap H_x) = \rho_F(I_k \cap H_c) + (F(c) - F(x))$$

and consequently $x \in A$, since

$$\begin{aligned}\sum_{j\in\mathbb{N}} \rho_F(I_j \cap H_x) &\geq \sum_{j\in\mathbb{N}} \rho_F(I_j \cap H_c) + (F(c) - F(x)) \\ &\geq (F(b) - F(c)) + (F(c) - F(x)) = F(b) - F(x).\end{aligned}$$

- But $x < c = \inf A$, which is impossible. Thus $a = c \in A$, so

$$\rho_F(I) = F(b) - F(a) \leq \sum_{j\in\mathbb{N}} \rho_F(I_j \cap H_c) \leq \sum_{j\in\mathbb{N}} \rho_F(I_j).$$

Proof

Suppose now that $\rho_F(I) = \infty$.

- Assume that $I = (a, \infty) = \bigcup_{n \in \mathbb{N}} I_n \in J_{oc}$ for some $a \in \mathbb{R}$. For any $n \in \mathbb{N}$ such that $n > a$ we have $(a, n] \subseteq (a, \infty)$ and

$$\rho_F((a, n]) \leq \rho_F((a, \infty))$$

then

$$F(n) - F(a) \leq \sum_{n \in \mathbb{N}} \rho_F(I_n).$$

- Passing with $n \to \infty$ we get

$$\rho_F(I) = \infty \leq \sum_{n \in \mathbb{N}} \rho_F(I_n).$$

This completes the proof of the theorem. □

Properties of the Lebesgue–Stieltjes measures

- The outer measure μ_F^* is induced by the premeasure $\mu_{F,0}$ on the algebra $\mathcal{A}$ of finite disjoint unions of elements from J_{oc}. The premeasure $\mu_{F,0}$ is σ-finite, since $\mathbb{R} = \bigcup_{n\in\mathbb{N}}(n, n+1]$.
- $(\mathbb{R}, \mathcal{L}_{\mu_F}, \mu_F)$ is a complete σ-finite measure space.
- By the Carathéodory extension theorem we conclude that
  1. $\mu_F(E) = \mu_F^*(E) = \mu_{F,0}(E)$ for any $E \in \mathcal{A}$;
  2. $\mathcal{A} \subseteq \sigma(J_{oc}) = \sigma(\mathcal{A}) = \mathrm{Bor}(\mathbb{R}) \subseteq \mathcal{L}_{\mu_F}$.
- Therefore it follows that μ_F is a σ-finite Borel measure and

$$\mu((a, b]) = F(b) - F(a)$$

for any $-\infty \le a \le b \le \infty$.

Properties of the Lebesgue–Stieltjes measures

- Moreover μ_F is unique in the sense that if G is another such function, we have $\mu_F = \mu_G$ if and only if $F - G$ is constant.
- Conversely, if μ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and we define

$$F(x) = \begin{cases} \mu((-\infty, x]) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu((x, 0]) & \text{if } x < 0, \end{cases}$$

then F is increasing and right continuous and $\mu = \mu_F$.