Lecture 11

Product measures

Graduate Real Analysis

Fall Semester

Measurable rectangles

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Our aim is to construct a measure on $\mathcal{M} \otimes \mathcal{N}$ that will be the product of μ and ν.

Measurable rectangle

A **measurable rectangle** is the set of the form

$$A \times B \in \mathcal{M} \times \mathcal{N} = \{A \times B : A \in \mathcal{M},\ B \in \mathcal{N}\}.$$

Remark

The collection $\mathcal{A}$ of finite disjoint unions of rectangles from $\mathcal{M} \times \mathcal{N}$ is an algebra, since $\mathcal{M} \times \mathcal{N}$ is a semi-algebra, namely

$$(A \times B) \cap (E \cap F) = (A \cap E) \times (B \cap F),$$
$$(A \times B)^c = (X \times B^c) \cup (A^c \times B).$$

Of course, $\mathcal{M} \otimes \mathcal{N} = \sigma(\mathcal{A}) = \sigma(\mathcal{M} \times \mathcal{N})$, but $\mathcal{M} \otimes \mathcal{N} \neq \mathcal{M} \times \mathcal{N}$.

Observation 1

Suppose $A \times B \in \mathcal{M} \times \mathcal{N}$ is a rectangle that is a (finite or countable) disjoint union of rectangles $A_j \times B_j \in \mathcal{M} \times \mathcal{N}$. Then for $(x, y) \in X \times Y$

$$\mathbb{1}_A(x)\mathbb{1}_B(y) = \mathbb{1}_{A\times B}((x,y)) = \sum_{j\in\mathbb{N}} \mathbb{1}_{A_j\times B_j}((x,y)) = \sum_{j\in\mathbb{N}} \mathbb{1}_{A_j}(x)\mathbb{1}_{B_j}(y),$$

If we integrate with respect to $x \in X$ and use the **(MCT)**, then

$$\begin{aligned}\mu(A)\mathbb{1}_B(y) &= \int_X \mathbb{1}_A(x)\mathbb{1}_B(y)\,d\mu(x)\\ &\underbrace{=}_{\text{(MCT)}} \sum_{j\in\mathbb{N}} \int_X \mathbb{1}_{A_j}(x)\mathbb{1}_{B_j}(y)\,d\mu(x) = \sum_{j\in\mathbb{N}} \mu(A_j)\mathbb{1}_{B_j}(y).\end{aligned}$$

If we integrate with respect to $y \in Y$ and use the **(MCT)** again, then

$$\mu(A)\nu(B) = \sum_{j\in\mathbb{N}} \mu(A_j)\nu(B_j).$$

From countably subadditive set functions to premeasures

Theorem (From countably subadditive set functions to premeasures)

Let $\rho : \mathcal{E} \to [0, \infty]$ be a set function on a semi-algebra $\mathcal{E}$ of subsets of a set X and $\rho(\emptyset) = 0$. Suppose that ρ is finitely additive on $\mathcal{E}$ and let $\mathcal{A}$ be the algebra that consists of all finite disjoint unions of members of $\mathcal{E}$. Define a set function μ_0 on $\mathcal{A}$ by setting

$$\mu_0(A) = \sum_{i=1}^{n} \rho(E_i)$$

for $A = \bigcup_{j=1}^{n} E_j \in \mathcal{A}$, where $E_1, \ldots, E_n \in \mathcal{E}$ and $E_i \cap E_j = \emptyset$ if $i \neq j$.

a. Then μ_0 is a well-defined additive set function on $\mathcal{A}$ such that $\mu_0(\emptyset) = 0$ and $\mu_0 = \rho$ on $\mathcal{E}$.
b. If additionally ρ is countably subadditive on $\mathcal{E}$ then μ_0 is a premeasure on $\mathcal{A}$.

Observation 2

- If
$$E = \bigcup_{j=1}^{n} A_j \times B_j \in \mathcal{A}$$
is the disjoint union of rectangles $A_1 \times B_1, \ldots, A_n \times B_n \in \mathcal{M} \times \mathcal{N}$ and we set
$$\pi(E) = \sum_{j=1}^{n} \mu(A_j)\nu(B_j)$$
with the usual convention that $0 \cdot \infty = 0$, then π is well defined on $\mathcal{A}$.

- Moreover, π is a premeasure on $\mathcal{A}$, since by Observation 1, it is countably additive on $\mathcal{M} \times \mathcal{N}$.

Product of two measures

Since π is the premeasure on $\mathcal{A}$, by the Caratheodory extension theorem, π generates an outer measure on $X \times Y$ whose extension to $\mathcal{M} \times \mathcal{N}$ is a measure that extends π.

Definition (Product measure)

We call this measure the **product of** μ **and** ν and denote it by $\mu \times \nu$.

Remark

- If μ and ν are σ-finite, then also $\mu \times \nu$ is σ-finite.
- In this case, $\mu \times \nu$ is the unique measure on $\mathcal{M} \otimes \mathcal{N}$ such that

$$\mu \times \nu(A \times B) = \mu(A)\nu(B)$$

for all rectangles $A \times B \in \mathcal{M} \times \mathcal{N}$.

Product of finitely many measures

- Suppose that $(X_j, \mathcal{M}_j, \mu_j)$ are measurable spaces $1 \leq j \leq n$. If we define a rectangle to be a set of the form

$$A_1 \times \ldots \times A_n \in \prod_{j=1}^{n} \mathcal{M}_j = \{A_1 \times \ldots \times A_n : A_1 \in \mathcal{M}_1, \ldots, A_n \in \mathcal{M}_n\},$$

then the collection $\mathcal{A}$ of finite disjoint unions of rectangles from $\prod_{j=1}^{n} \mathcal{M}_j$ is an algebra.

- This algebra gives rise to the canonical premeasure as in the case of two factors, which produces a **product measure**

$$\mu_1 \times \ldots \times \mu_n \quad \text{on} \quad \mathcal{M}_1 \otimes \ldots \otimes \mathcal{M}_n$$

such that

$$\mu_1 \times \ldots \times \mu_n(A_1 \times \ldots \times A_n) = \prod_{j=1}^{n} \mu_j(A_j),$$

for any $A_1 \times \ldots \times A_n \in \prod_{j=1}^{n} \mathcal{M}_j$.

Remarks

Remarks

- If the μ_j's are σ-finite so that the extension from $\mathcal{A}$ to $\bigotimes_{j=1}^{n} \mathcal{M}_j$ is uniquely determined by the Caratheodory extension theorem.
- Moreover, the obvious associativity holds. For example, if we identify

$$X_1 \times X_2 \times X_3 \quad \text{with} \quad (X_1 \times X_2) \times X_3$$

we have $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3 = (\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3$, where

  - $\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{M}_3 = \sigma(\mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3)$ and
  - $(\mathcal{M}_1 \otimes \mathcal{M}_2) \otimes \mathcal{M}_3 = \sigma(\mathcal{N} \times \mathcal{M}_3)$, with $\mathcal{N} = \mathcal{M}_1 \otimes \mathcal{M}_2$,

and we also have

$$\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3,$$

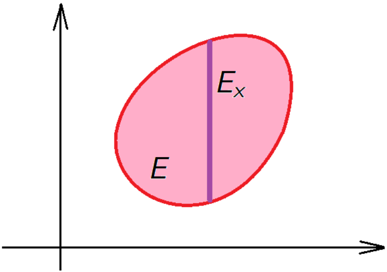
since they agree on the sets of the form $A_1 \times A_2 \times A_3$ and hence in general by the uniqueness.

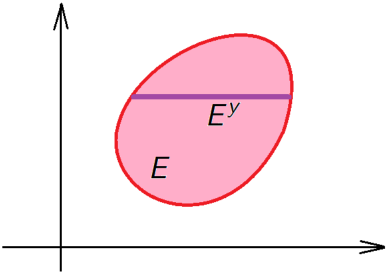
Sections

Definition (Sections)

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. If $E \in X \times Y$, for $x \in X$ and $y \in Y$ we define the x-**section** of E and y-**section** of E by

$$E_x = \{y \in Y : (x, y) \in E\}, \qquad \text{and} \qquad E^y = \{x \in X : (x, y) \in E\}.$$





Sections of functions

Definition (Sections of functions)

If f is a function on $X \times Y$ we define the x-**section** f_x and the y-**section** of f by

$$f_x(y) = f^y(x) = f(x, y).$$

Example

$$(\mathbb{1}_E)_x = \mathbb{1}_{E_x}, \qquad \text{and} \qquad (\mathbb{1}_E)^y = \mathbb{1}_{E^y}.$$

Proposition

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces.

1. *If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.*
2. *If f is $\mathcal{M} \otimes \mathcal{N}$-measurable then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$.*

Proof

To prove the conclusion from the first item let

$$\mathcal{R} = \{E \subseteq X \times Y : E_x \in \mathcal{N} \text{ for all } x \in X, \quad E^y \in \mathcal{M} \text{ for all } y \in Y\}.$$

Then $\mathcal{M} \times \mathcal{N} \subseteq R$, since

$$(A \times B)_x = \begin{cases} B & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A, \end{cases} \quad \text{and} \quad (A \times B)^y = \begin{cases} A & \text{if } y \in B, \\ \emptyset & \text{if } y \notin B. \end{cases}$$

Since

$$\left(\bigcup_{j=1}^{\infty} E_j\right)_x = \bigcup_{j=1}^{\infty} (E_j)_x, \quad \text{and} \quad (E^c)_x = E_x^c,$$

and likewise for y-sections, thus $\mathcal{R}$ is a σ-algebra and consequently $\mathcal{R} \supseteq \mathcal{M} \otimes \mathcal{N}$. For the second item we use the first one and note that

$$(f_x)^{-1}[B] = \{y \in Y : f_x(y) \in B\} = \{y \in Y : f(x, y) \in B\} = (f^{-1}[B])_x$$

and similarly $(f^y)^{-1}[B] = (f^{-1}[B])^y$. $\square$

Monotone classes

Definition (Monotone class)

A monotone class $\mathcal{C}$ on a space X is a subset of $\mathcal{P}(X)$ that is closed under countable increasing unions, that is

- if $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ and $C_1 \subseteq C_2 \subseteq \ldots$, then $\bigcup_{n\in\mathbb{N}} C_n \in \mathcal{C}$,

and countable decreasing intersections, that is

- if $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$ and $C_1 \supseteq C_2 \supseteq \ldots$, then $\bigcap_{n\in\mathbb{N}} C_n \in \mathcal{C}$.

Remark

- Clearly, every σ-algebra is a monotone class.
- Intersection of any family of monotone classes is a monotone class.
- Intersection of all monotone classes containing $\mathcal{E} \subseteq \mathcal{P}(X)$ is a unique smallest monotone class containing $\mathcal{E}$.

The monotone class lemma

Lemma

If $\mathcal{A}$ is an algebra of subsets of X then the monotone class $\mathcal{C}$ generated by $\mathcal{A}$ coincides with the σ-algebra $\mathcal{M} = \sigma(\mathcal{A})$.

Proof. Since $\mathcal{M}$ is a monotone class, we have $\mathcal{C} \subseteq \mathcal{M}$. We show that $\mathcal{C}$ is a σ-algebra, yielding $\mathcal{M} \subseteq \mathcal{C}$. For $E \in \mathcal{C}$ let us define a monotone class

$$\mathcal{C}(E) = \{F \in \mathcal{C} : E \setminus F \text{ and } F \setminus E \text{ and } E \cap F \in \mathcal{C}\}.$$

- Clearly, $\emptyset, E \in \mathcal{C}(E)$, since $\mathcal{A} \subseteq \mathcal{C}$, and $F \in \mathcal{C}(E)$ iff $E \in \mathcal{C}(F)$.
- If $E \in \mathcal{A}$, then $F \in \mathcal{C}(E)$ for all $F \in \mathcal{A}$ because $\mathcal{A} \subseteq \mathcal{C}$ is an algebra. Thus $\mathcal{A} \subseteq \mathcal{C}(E)$ and $\mathcal{C} \subseteq \mathcal{C}(E)$ for all $E \in \mathcal{A}$.
- If $F \in \mathcal{C}$, then $F \in \mathcal{C}(E)$ for all $E \in \mathcal{A}$. Then $E \in \mathcal{C}(F)$ for all $E \in \mathcal{A}$, so that $\mathcal{A} \subseteq \mathcal{C}(F)$ and hence $\mathcal{C} \subseteq \mathcal{C}(F)$.
- If $E, F \in \mathcal{C}$, then $F \setminus E, E \cap F \in \mathcal{C}$, since $X \in \mathcal{A} \subseteq \mathcal{C}$ hence $\mathcal{C}$ is an algebra. If $(E_j)_{j \in \mathbb{N}} \subseteq \mathcal{C}$ then $\bigcup_{j=1}^{n} E_j \in \mathcal{C}$ but $\mathcal{C}$ is closed under countable increasing unions thus $\mathcal{C}$ is a σ-algebra as claimed. $\square$

Fundamental Theorem

Theorem (Fundamental Theorem)

Suppose $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are* σ*-finite measure spaces. If* $E \in \mathcal{M} \otimes \mathcal{N}$ *then the functions*

$$x \mapsto \nu(E_x) \qquad \text{and} \qquad y \mapsto \mu(E^y)$$

are measurable on X *and* Y *respectively and*

$$\mu \times \nu(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y).$$

Proof. First suppose that μ and ν are finite. Let $\mathcal{C}$ be the collection of all $E \in \mathcal{M} \otimes \mathcal{N}$ for which the conclusion of the theorem is true. If $E = A \times B$ then $\nu(E_x) = \mathbb{1}_A(x)\nu(B)$ and $\mu(E^y) = \mu(A)\mathbb{1}_B(y)$ thus $E \in \mathcal{C}$. By additivity it follows that finite disjoint unions of rectangles are in $\mathcal{C}$. It suffices to show that $\mathcal{C}$ is a monotone class and then the conclusion will follow by the monotone class lemma.

Proof

- If $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{C}$ and
$$E = \bigcup_{j=1}^{\infty} E_j,$$
then the functions $f_n(y) = \mu((E_n)^y)$ are measurable and increase pointwise to $f(y) = \mu(E^y)$.
- Hence f is measurable and by the **(MCT)** we see
$$\int_Y \mu(E^y)d\nu(y) = \lim_{n\to\infty}\int_Y \mu(E_n^y)d\nu(y) = \lim_{n\to\infty}\mu\times\nu(E_n) = \mu\times\nu(E).$$
- Likewise $\mu\times\nu(E) = \int_X \nu(E_x)d\mu(x)$ so $E\in\mathcal{C}$.

Proof

- Similarly, if $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{C}$ and $\bigcap_{n=1}^{\infty} E_n = F$, the function
$$y \mapsto \mu((E_1)^y) \in L^1(\nu)$$
because $\mu((E_1)^y) \leq \mu(X) < \infty$ and $\nu(Y) < \infty$, so the **(DCT)** can be applied to show that $F \in \mathcal{C}$.
- Thus $\mathcal{C}$ is a monotone class for the case of finite measure spaces.
- If μ and ν are σ-finite, then $X \times Y = \bigcup_{j\in\mathbb{N}} X_j \times Y_j$ for an increasing sequence of rectangles $(X_j \times Y_j)_{j\in\mathbb{N}}$ with $\mu(X_j \times Y_j) < \infty$ for $j \in \mathbb{N}$.
- If $E \in \mathcal{M} \otimes \mathcal{N}$ then invoking the finite measure case we see that
$$\begin{aligned} \mu \times \nu(E \cap (X_j \times Y_j)) &= \int_X \mathbb{1}_{X_j}(x)\nu(E_x \cap Y_j)d\mu(x) \\ &= \int \mathbb{1}_{Y_j}(y)\mu(E^y \cap X_j)d\nu(y). \end{aligned}$$

By the **(MCT)** the claim follows for σ-finite measure spaces. $\square$

Tonelli's Theorem

Theorem (Tonelli's Theorem)

Suppose that $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are* σ*-finite measure spaces. If* $f \in L^0_+(X \times Y, \mu \times \nu)$ *then*

$$g(x) = \int_Y f_x(y)d\nu(y) \in L^0_+(X, \mu),$$
$$h(y) = \int_X f^y(x)d\mu(x) \in L^0_+(Y, \nu),$$

and

$$\int_{X\times Y} f(x,y)d(\mu \times \nu)(x,y) = \int_X \left(\int_Y f(x,y)d\nu(y)\right) d\mu(x)$$
$$= \int_Y \left(\int_X f(x,y)d\mu(x)\right) d\nu(y).$$

Proof

- Tonelli's theorem reduces to the previous theorem in case f is a characteristic function and therefore holds for nonnegative simple functions by linearity. If $f \in L^0_+(X \times Y)$ let $(f_n)_{n\in\mathbb{N}}$ be a sequence of simple functions that increase pointwise to f. Let

$$g_n(x) = \int_Y (f_n)_x(y)d\nu(y), \quad \text{and} \quad h_n(y) = \int_X (f_n)^y(x)d\mu(x).$$

- By the **(MCT)** $g_n \xrightarrow[n\to\infty]{} g$ and $h_n \xrightarrow[n\to\infty]{} h$, so that g and h are measurable and we have

$$\int_X g d\mu = \lim_{n\to\infty} \int_X g_n d\mu = \lim_{n\to\infty} \int_{X\times Y} f_n d(\mu \times \nu) = \int_{X\times Y} f d(\mu \times \nu),$$
$$\int_Y h d\nu = \lim_{n\to\infty} \int_Y h_n d\nu = \lim_{n\to\infty} \int_{X\times Y} f_n d(\mu \times \nu) = \int_{X\times Y} f d(\mu \times \nu).$$

This proves Tonelli's theorem. $\square$

Fubini's Theorem

Theorem (Fubini's Theorem)

Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $f \in L^1(X \times Y, \mu \times \nu)$ then $f_x \in L^1(Y, \nu)$ for μ-a.e. $x \in X$ and $f^y \in L^1(X, \mu)$ for ν-a.e. $y \in Y$ and

$$g(x) = \int_Y f_x(y) d\nu(y) \in L^1(X, \mu),$$
$$h(y) = \int_X f^y(x) d\mu(x) \in L^1(Y, \nu),$$

and

$$\int_{X\times Y} f(x,y) d(\mu \times \nu)(x,y) = \int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x)$$
$$= \int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y).$$

Proof

- In Tonelli's theorem if
$$\int_{X\times Y} f d\mu \times \nu < \infty,$$
then
$$g(x) = \int_Y f_x(y) d\nu(y) < \infty, \quad \mu - \text{a.e.},$$
$$h(y) = \int_X f^y(x) d\mu(x) < \infty, \quad \nu - \text{a.e.},$$
that is $f_x \in L^1(Y,\nu)$ for a.e. $x \in X$ and $f^y \in L^1(X,\mu)$ for a.e. $y \in Y$.
- If $f \in L^1(X \times Y, \mu \times \nu)$ then the conclusion of Fubini's theorem follows by applying Tonelli's theorem and these results to the positive and negative parts of the real and imaginary parts of f. $\square$

Remarks

- σ-finiteness is necessary in both Tonelli's and Fubini's theorems.
- The Fubini and Tonelli theorems are frequently used in tandem. To reverse the order of integration in a double integral $\int_{X\times Y} fd\mu d\nu$, we first verify that $\int_{X\times Y} |f| d\mu d\nu < \infty$ by using Tonelli's theorem to evaluate this integral as an iterated integral; then one applies Fubini's theorem to conclude that $\int_Y \int_X fd\mu d\nu = \int_X \int_Y fd\nu d\mu$.
- If μ and ν are complete $\mu \times \nu$ is almost never complete. Suppose that there is $A \in \mathcal{M}$ and $\mu(A) = 0$ and $\mathcal{N} \neq \mathcal{P}(Y)$ (think that $\mu = \nu = \lambda$ is the Lebesgue measure on $\mathbb{R}$). If $E \in \mathcal{P}(Y) \setminus \mathcal{N}$, then $A \times E \notin \mathcal{M} \otimes \mathcal{N}$ but $A \times E \subseteq A \times Y$ and $\mu \times \nu(A \times Y) = 0$.
- If one wishes to work with complete measures, of course, one can consider the completion of $\mu \times \nu$. In this setting the relationship between the measurability of a function on $X \times Y$ and the measurability of its x-sections and y-sections is not so simple. However, the Fubini–Tonelli theorem is still valid when suitably reformulated.

Completion of product spaces

Theorem

For a given $d \in \mathbb{N}$ let $(X_j, \mathcal{M}_j, \mu_j)$ be a measure space for each $1 \le j \le d$. Let $(X_j, \overline{\mathcal{M}_j}, \overline{\mu_j})$ be a completion of $(X_j, \mathcal{M}_j, \mu_j)$. Then one has

$$\left(\prod_{j=1}^{d} X_j, \overline{\sigma\left(\prod_{j=1}^{d} \mathcal{M}_j\right)}, \overline{\bigotimes_{j=1}^{d} \mu_j}\right) = \left(\prod_{j=1}^{d} X_j, \overline{\sigma\left(\prod_{j=1}^{d} \overline{\mathcal{M}_j}\right)}, \overline{\bigotimes_{j=1}^{d} \overline{\mu_j}}\right).$$

Proof. If $(X, \mathcal{M}, \mu)$ is a measure space, then

$$\overline{\mathcal{M}} = \{A \cup B : A \in \mathcal{M} \text{ and } B \subseteq C \in \mathcal{M} \text{ and } \mu(C) = 0\}.$$

Since $\mathcal{M}_j \subseteq \overline{\mathcal{M}_j}$ for each $1 \le j \le d$, then

$$\sigma\left(\prod_{j=1}^{d} \mathcal{M}_j\right) \subseteq \sigma\left(\prod_{j=1}^{d} \overline{\mathcal{M}_j}\right) \quad \Longrightarrow \quad \overline{\sigma\left(\prod_{j=1}^{d} \mathcal{M}_j\right)} \subseteq \overline{\sigma\left(\prod_{j=1}^{d} \overline{\mathcal{M}_j}\right)}.$$

Proof

It suffices to show that

$$\prod_{j=1}^{d} \overline{\mathcal{M}_j} \subseteq \overline{\sigma\Big(\prod_{j=1}^{d} \mathcal{M}_j\Big)} \quad \Longrightarrow \quad \overline{\sigma\Big(\prod_{j=1}^{d} \overline{\mathcal{M}_j}\Big)} \subseteq \overline{\sigma\Big(\prod_{j=1}^{d} \mathcal{M}_j\Big)}.$$

- If $E_j \in \overline{\mathcal{M}_j}$ then $E_j = A_j \cup B_j$, where $A_j \in \mathcal{M}_j$ and $B_j \subseteq C_j \in \mathcal{M}_j$ and $\mu_j(C_j) = 0$. Define

$$P_j = F_{j,1} \times \ldots \times F_{j,d} \quad \text{where} \quad F_{j,k} = \begin{cases} A_k \cup B_k & \text{for } k \neq j, \\ B_j & \text{for } k = j, \end{cases}$$

and

$$Q_j = G_{j,1} \times \ldots \times G_{j,d} \quad \text{where} \quad G_{j,k} = \begin{cases} A_k \cup C_k & \text{for } k \neq j, \\ C_j & \text{for } k = j. \end{cases}$$

Proof

- Then
$$\prod_{j=1}^{d} E_j = \prod_{j=1}^{d}(A_j \cup B_j) = \prod_{j=1}^{d} A_j \cup \bigcup_{j=1}^{d} P_j.$$
Now $\prod_{j=1}^{d} A_j \in \prod_{j=1}^{d} \mathcal{M}_j$ and $\bigcup_{j=1}^{d} P_j \subseteq \bigcup_{j=1}^{d} Q_j \in \prod_{j=1}^{d} \mathcal{M}_j$ and
$$\Big(\bigotimes_{k=1}^{d} \mu_k\Big)\Big(\bigcup_{j=1}^{d} Q_j\Big) \leq \sum_{j=1}^{d} \Big(\bigotimes_{k=1}^{d} \mu_k\Big)(Q_j) \leq \sum_{j=1}^{d} \prod_{k=1}^{d} \mu_k(G_{j,k}) = 0,$$
since $\mu_k(G_{k,k}) = \mu_k(C_k) = 0$.
- Thus we have shown that $\bigcup_{j=1}^{d} P_j$ is a subset of a null set $\bigcup_{j=1}^{d} Q_j$ in a product space $\left(\prod_{j=1}^{d} X_j, \sigma\left(\prod_{j=1}^{d} \mathcal{M}_j\right), \bigotimes_{j=1}^{d} \mu_j\right)$, as desired. □

Observation

Lemma

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be two measure spaces. If E is a null set in the product space $(X \times Y, \sigma(\mathcal{M} \times \mathcal{N}), \mu \times \nu)$ then

- *E_x is a null set in $(Y, \mathcal{N}, \nu)$ for μ-a.e. $x \in X$.*
- *E^y is a null set in $(X, \mathcal{M}, \mu)$ for ν-a.e. $y \in Y$.*

Proof. We have

$$\int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y) = \mu \times \nu(E) = 0.$$

Thus

$$\nu(E_x) \geq 0 \quad \text{and} \quad \int_X \nu(E_x) d\mu(x) = 0,$$

yielding $\nu(E_x)$ for μ-a.e. $x \in X$. Similarly, $\mu(E^y)$ for ν-a.e. $y \in Y$. $\square$

Useful fact

Proposition

Let $(X \times Y, \overline{\sigma(\mathcal{M} \times \mathcal{N})}, \mu \times \nu)$ be the completion of the product space $(X \times Y, \sigma(\mathcal{M} \times \mathcal{N}), \mu \times \nu)$ of two complete measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$.

(a) *If $E \in \overline{\sigma(\mathcal{M} \times \mathcal{N})}$, then $E_x \in \mathcal{N}$ for μ-a.e. $x \in X$ and $E^y \in \mathcal{M}$ for ν-a.e. $y \in Y$.*

(b) *If f is an extended real-valued $\overline{\sigma(\mathcal{M} \times \mathcal{N})}$-measurable function on $X \times Y$, then f_x is $\mathcal{N}$-measurable for μ-a.e. $x \in X$ and f^y is $\mathcal{M}$-measurable for ν-a.e. $y \in Y$.*

Proof. It suffices to prove (b), the (a) will follow by taking $f = \mathbb{1}_E$ for $E \in \overline{\sigma(\mathcal{M} \times \mathcal{N})}$. An important tool in proving (b) is the following result:

Proof

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measurable space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is $\overline{\mathcal{M}}$-measurable function on X, there is $\mathcal{M}$-measurable function g such that $f = g$ $\overline{\mu}$-a.e., in fact $f = g$ on N^c for some null set $N \in \mathcal{M}$.

If f is $\overline{\sigma(\mathcal{M} \times \mathcal{N})}$-measurable then there are a $\sigma(\mathcal{M} \times \mathcal{N})$-measurable function g and a null set $E \in \sigma(\mathcal{M} \times \mathcal{N})$ such that $f = g$ on E^c. Then

- $f_x = g_x$ on E_x^c for $x \in X$,
- $f^y = g^y$ on $(E^y)^c$ for $y \in Y$.

Now g_x is $\mathcal{N}$-measurable for every $x \in X$. By the previous observation E_x is a null set in a complete measure space $(Y, \mathcal{N}, \nu)$ for μ-a.e. $x \in X$. Thus f_x is $\mathcal{N}$-measurable for μ-a.e. $x \in X$. Similarly, we have f^y is $\mathcal{M}$-measurable for ν-a.e. $y \in Y$, and the proof is completed. □

- Using this proposition and previous observation we can readily prove the following Fubini–Tonelli theorem for complete measure spaces.

Fubini–Tonelli theorem for complete measure spaces

Theorem (Fubini–Tonelli theorem for complete measure spaces)

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be complete σ-finite measure spaces and let $(X \times Y, \mathcal{T}, \theta)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{T}$-measurable and either: (a) $f \geq 0$, or (b) $f \in L^1(\theta)$. Then f_x is $\mathcal{N}$-measurable for a.e. x and f^y is $\mathcal{M}$-measurable for a.e. y., and in case (b) f_x and f^y are also integrable for a.e. x and y. Moreover,

$$g(x) = \int_Y f_x(y) d\nu(y), \quad \text{and} \quad h(y) = \int_X f^y(x) d\mu(x),$$

are measurable and in case (b) also integrable, and

$$\begin{aligned}\int_{X \times Y} f(x,y) d\theta(x,y) &= \int_X \left(\int_Y f(x,y) d\nu(y) \right) d\mu(x) \\ &= \int_Y \left(\int_X f(x,y) d\mu(x) \right) d\nu(y).\end{aligned}$$