Lecture 12

Change of variables formula
Spherical measure and polar coordinates

Graduate Real Analysis

Fall Semester

Euclidean spaces

Let $\mathbb{R}^d$ be the Euclidean d-dimensional space with the Euclidean norm

$$|x| = \left(\sum_{j=1}^{d} |x_j|^2 \right)^{1/2} = \sqrt{\langle x, x \rangle},$$

which is induced by the standard inner product

$$\langle x, y \rangle = x \cdot y = \left(\sum_{j=1}^{d} x_j y_j \right)^{1/2},$$

for any $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{R}^d$.

Jacobians

Jacobians

If $E \subseteq \mathbb{R}^d$ is open and $f : E \to \mathbb{R}^d$ is differentiable at a point $x \in E$, the determinant of the linear operator $D_x f = f'(x)$ is called the **Jacobian** of f at $x \in \mathbb{R}^d$ and denoted by

$$J_f(x) = \det D_x f = \det f'(x).$$

We shall also use the notation

$$\frac{\partial(y_1, \ldots, y_d)}{\partial(x_1, \ldots, x_d)} = J_f(x) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial x_1} & \cdots & \frac{\partial f_d}{\partial x_d} \end{bmatrix}.$$

if $(y_1, \ldots, y_d) = (f_1(x_1, \ldots, x_d), \ldots, f_d(x_1, \ldots, x_d)) = f(x_1, \ldots, x_d)$.

Some comments

- In terms of Jacobians, the crucial hypothesis in the inverse function theorem is that $J_f(x) \neq 0$.
- We will investigate behavior of the Lebesgue integral under linear transformations. We will identify a linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ with the matrix
$$(T_{ij})_{1\le i,j\le d} = (e_i \cdot Te_j)_{1\le i,j\le d},$$
where $\{e_j : 1 \le j \le d\}$ is the standard basis in $\mathbb{R}^d$.
- We denote the determinant of this matrix by $\det T$ and recall that
$$\det(T \circ S) = (\det T)(\det S).$$
- Furthermore, $\mathrm{GL}(d, \mathbb{R})$ denotes the general linear group of invertible linear transformations of $\mathbb{R}^d$.

Transformations of elementary types

Recall that every $T \in \mathrm{GL}(d, \mathbb{R})$ can be written as the product of finitely many transformations of three elementary types:

1. $$T_1(x_1, \ldots, x_j, \ldots, x_d) = (x_1, \ldots, cx_j, \ldots, x_d),$$

2. $$T_2(x_1, \ldots, x_j, \ldots, x_d) = (x_1, \ldots, x_j + cx_k, \ldots, x_d), \quad \text{for} \quad k \neq j,$$

3. $$T_3(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_d) = (x_1, \ldots, x_k, \ldots, x_j, \ldots, x_d).$$

That every $T \in \mathrm{GL}(d, \mathbb{R})$ is a product of transformations of these three types is simply the fact that every invertible matrix can be row-reduced to the identity matrix.

Change of variables for linear transformations

From now on we shall consider $\mathbb{R}^d$ with Lebesgue measure λ_d either on $\mathcal{L}(\mathbb{R}^d)$ or on $\mathrm{Bor}(\mathbb{R}^d) = \mathcal{B}(\mathbb{R}^d)$. We shall also abbreviate $d\lambda_d(x)$ to dx.

Theorem

Suppose $T \in \mathrm{GL}(d, \mathbb{R})$.

(a) *If f is a Lebesgue measurable function on $\mathbb{R}^d$, so is $f \circ T$. If $f \geq 0$ or $f \in L^1(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} f(x)dx = |\det T| \int_{\mathbb{R}^d} f \circ T(x)dx. \qquad (*)$$

(b) *If $E \in \mathcal{L}(\mathbb{R}^d)$, then $T[E] \in \mathcal{L}(\mathbb{R}^d)$ and*

$$\lambda_d(T[E]) = |\det T| \lambda_d(E).$$

Proof

Proof. First suppose that f is Borel measurable, then $f \circ T$ is Borel measurable since T is continuous.

- If (*) is true for $T, S \in \mathrm{GL}(d, \mathbb{R})$ it is also true for $T \circ S$ since
$$\int_{\mathbb{R}^d} f(x)dx = |\det T| \int_{\mathbb{R}^d} f(Tx)dx = |\det T||\det S| \int_{\mathbb{R}^d} f(TSx)dx.$$
It suffices to prove (*) when $T \in \{T_1, T_2, T_3\}$ described above, which will be a simple consequence of the Fubini–Tonelli theorem.
- Indeed, for T_3 we interchange the order of integration in the variables x_j and x_k, and for T_1 and T_2 we integrate first with respect to x_j and use the one-dimensional formulas
$$\int_{\mathbb{R}} f(t)dt = |c| \int_{\mathbb{R}} f(xt)dt, \quad \text{and} \quad \int_{\mathbb{R}} f(t+a)dt = \int_{\mathbb{R}} f(t)dt.$$
- It is easily verified that $\det T_1 = c$, $\det T_2 = 1$, $\det T_3 = -1$, so the formula (*) is proved.

Proof

- Moreover, if E is a Borel set, then $\lambda_d(T[E]) = |\det T|\lambda_d(E)$ by taking $f = \mathbb{1}_{T[E]}$ in (*) since $T[E]$ is a Borel set as well. WHY? In particular, the class of Borel null sets is invariant under T and T^{-1}.
- Suppose $E \in \mathcal{L}(\mathbb{R}^d)$, since $\mathcal{L}(\mathbb{R}^d)$ is a completion of $\mathcal{B}(\mathbb{R}^d)$, thus $E = A \cup B$, where $A \in \mathcal{L}(\mathbb{R}^d)$ and $B \subseteq C \in \mathcal{B}(\mathbb{R}^d)$ and $\lambda_d(C) = 0$.
- Then $\lambda_d(T[B]) \le \lambda_d(T[C]) = |\det T|\lambda_d(C) = 0$ and $T[B] \in \mathcal{L}(\mathbb{R}^d)$, since λ_d is complete. Thus

$$\lambda_d(T[E]) = \lambda_d(T[A]) = |\det T|\lambda_d(A) = |\det T|\lambda_d(E),$$

since $\lambda_d(T[E]) = \lambda_d(T[A] \cup T[B]) = \lambda_d(T[A])$ and $\lambda_d(A) = \lambda_d(E)$.
- By the previous argument (*) holds for $f = \mathbb{1}_E$ if $E \in \mathcal{L}(\mathbb{R}^d)$. By linearity (*) is true for step functions, by the **(MTC)** is true for nonnegative $\mathcal{L}(\mathbb{R}^d)$-measurable functions. Consequently, (*) remains true for all $\mathcal{L}(\mathbb{R}^d)$-measurable functions by considering positive and negative parts of the real and imaginary parts of f. $\square$

Our goal

Corollary

Lebesgue measure is invariant under rotations.

Proof. $T \in \mathrm{GL}(d, \mathbb{R})$ is a rotation iff $TT^* = I$, where T^* is the transpose of T. Since $\det T = \det T^*$ we obtain $|\det T| = 1$. $\square$

We now generalize previous theorem to differentiable maps. Let $\Omega \subseteq \mathbb{R}^d$ be an open set and $G = (g_1, \ldots, g_n) : \Omega \to \mathbb{R}^d$ be a map with $g_j \in C^1(\Omega)$.

Definition (Diffeomorphism)

$G : \Omega \to \mathbb{R}^d$ is called a **diffeomorphism** if G is injective and $D_x G$ is invertible for all $x \in \Omega$. In this case, the inverse function theorem guarantees that $G^{-1} : G[\Omega] \to \Omega$ is also a C^1 diffeomorphism and that

$$D_x(G^{-1}) = (D_{G^{-1}(x)} G)^{-1} \quad \text{for all} \quad x \in G[\Omega].$$

Change of variables for diffeomorphisms

Theorem

Let $\Omega \subseteq \mathbb{R}^d$ be open and let $G : \Omega \to \mathbb{R}^d$ be a $C^1(\Omega)$ diffeomorphism.

- (a) *If f is a Lebesgue measurable function on $G[\Omega]$ then $f \circ G$ is Lebesgue measurable on Ω. If $f \geq 0$ or $f \in L^1(G[\Omega], \lambda_d)$, then*

$$\int_{G[\Omega]} f(x)dx = \int_{\Omega} f \circ G(x)|\det D_x G|dx.$$

- (b) *If $E \subseteq \Omega$ and $E \in \mathcal{L}(\mathbb{R}^d)$ then $G[E] \in \mathcal{L}(\mathbb{R}^d)$ and*

$$\lambda_d(G[E]) = \int_E |\det D_x G|dx.$$

Notation

Notation

For $x \in \mathbb{R}^d$ and $T = (T_{ij})_{1 \leq i,j \leq d} \in \mathrm{GL}(d, \mathbb{R})$ we set

$$|x|_\infty = \max_{1 \leq j \leq d} |x_j|, \qquad \|T\| = \max_{1 \leq i \leq d} \sum_{j=1}^{d} |T_{ij}|.$$

Then it is easy to see that

$$|Tx|_\infty \leq \|T\| |x|_\infty$$

and

$$a + [-r, r]^d = \{x \in \mathbb{R}^d : |x - a|_\infty \leq r\},$$

is the cube of side length $2r$ centered at $a \in \mathbb{R}^d$.

Proof

Proof. It suffices to consider Borel measurable functions and sets. Since G and G^{-1} are both continuous, there are no problems with measurability.

- Let $Q = \{x \in \mathbb{R}^d : |x - a|_\infty \leq r\} \subseteq \Omega$. By the mean value theorem:

$$g_j(x) - g_j(a) = \sum_{i=1}^{d} (x_i - a_i) \frac{\partial g_j}{\partial x_i}(y)$$

  for some y lying on the the line segment joining x and a, so that

$$|G(x) - G(a)|_\infty \leq r\Big(\sup_{y \in Q} \|D_y G\|\Big) \quad \text{for} \quad x \in Q.$$

- In other words, $G[Q]$ is contained in the cube of side length $\sup_{y \in Q} \|D_y G\|$ times that of Q, so by the previous theorem,

$$\lambda_d(G[Q]) \leq \Big(\sup_{y \in Q} \|D_y G\|\Big)^d \lambda_d(Q).$$

Proof

- If $T \in \mathrm{GL}(d, \mathbb{R})$ we can apply this formula with G replaced by $T^{-1} \circ G$ together with the previous theorem to obtain

$$\begin{aligned} \lambda_d(G[Q]) &= |\det T| \lambda_d(T^{-1}(G[Q])) \\ &\le |\det T| \left(\sup_{y \in Q} \|T^{-1} D_y G\| \right)^d \lambda_d(Q). \qquad (**) \end{aligned}$$

- Since $y \mapsto D_y G$ is continuous, for any $\varepsilon > 0$ we choose $\delta > 0$ so that

$$\|(D_z G)^{-1}(D_y G)\|^d \le 1 + \varepsilon$$

  if $y, z \in Q$ and $|y - z|_\infty \le \delta$.
- Let us subdivide Q into subcubes $Q_1, \ldots, Q_N$ centered respectively at $x_1, \ldots, x_N$, whose interiors are disjoint, and whose side lengths are at most δ.

Proof

- By (**) with Q_j in place of Q and $T = D_x G$ we see that

$$\begin{aligned}\lambda_d(G[Q]) &\leq \sum_{j=1}^{N} \lambda_d(G[Q_j]) \\ &\leq \sum_{j=1}^{N} |\det D_{x_j} G| \Big(\sup_{y \in Q_j} \|(D_{x_j} G)^{-1} D_y G\| \Big)^d \lambda_d(Q_j) \\ &\leq (1+\varepsilon) \sum_{j=1}^{N} |\det D_{x_j} G| \lambda_d(Q_j).\end{aligned}$$

- This last sum is the integral of

$$\sum_{j=1}^{N} |\det D_{x_j} G| \mathbb{1}_{Q_j}(x).$$

Proof

- Then we see that

$$\sum_{j=1}^{N} |\det D_{x_j} G| \mathbb{1}_{Q_j} \xrightarrow[\delta \to 0]{} |\det D_x G| \mathbb{1}_Q$$

  converges uniformly, since $D_x G$ is continuous.
- Thus letting $\delta \to 0$ and $\varepsilon \to 0$ we find that

$$\lambda_d(G[Q]) \leq \int_Q |\det D_x G| dx. \qquad (***)$$

- We claim that (***) holds with Q replaced by any Borel set in Ω. Indeed, if $U \subseteq \Omega$ is open we can write

$$U = \bigcup_{j=1}^{\infty} Q_j,$$

  where Q_j's are cubes with disjoint interiors.

Proof

- Since the boundaries of the cubes have Lebesgue measure zero, hence by (***) we have

$$\lambda_d(G[U]) = \sum_{j=1}^{\infty} \lambda_d(G[Q_j]) \leq \sum_{j=1}^{\infty} \int_{Q_j} |\det D_x G| dx = \int_U |\det D_x G| dx.$$

- Next, let

$$W_k = \{x \in \Omega : |x|_\infty < k \text{ and } |\det D_x G| < k\},$$

and note that $W_k \subseteq W_{k+1}$ for $k \in \mathbb{N}$.

- If E is a Borel subset of W_k there is a decreasing sequence of open sets $U_j \subseteq W_{k+1}$ such that

$$E \subseteq \bigcap_{j=1}^{\infty} U_j \quad \text{and} \quad \lambda_d\Big(\bigcap_{j=1}^{\infty} U_j \setminus E\Big) = 0.$$

Proof

- By the proceeding estimate and the dominated convergence theorem we have

$$\lambda_d(G[E]) \leq \lambda_d\left(G\left[\bigcap_{j=1}^{\infty} U_j\right]\right) = \lim_{j\to\infty} \lambda_d(G[U_j])$$
$$\leq \lim_{j\to\infty} \int_{U_j} |\det D_x G| dx = \int_E |\det D_x G| dx.$$

- Finally, if E is any Borel subset of Ω we apply this argument to $E \cap W_k$ let $k \to \infty$ and conclude via the monotone convergence theorem that
$$\lambda_d(G[E]) \leq \int_E |\det D_x G| dx.$$

Proof

- If $f = \sum_{j=1}^{M} a_j \mathbb{1}_{A_j}$ is a nonnegative simple function on $G[\Omega]$ we therefore have

$$\int_{G[\Omega]} f(x)dx = \sum_{j=1}^{M} a_j \lambda_d(A_k) \leq \sum_{j=1}^{M} a_j \int_{G[A_j]} |\det D_x G| dx$$
$$= \int_{\Omega} f \circ G(x) |\det D_x G| dx.$$

- By the monotone convergence we conclude that the same is true for any nonnegative measurable function f, i.e.

$$\int_{G[\Omega]} f(x)dx \leq \int_{\Omega} f \circ G(x) |\det D_x G| dx.$$

Proof

- The same reasoning applied with G replaced by G^{-1} and f replaced by $f \circ G$, yields that

$$\begin{aligned}\int_\Omega f \circ G(x) |\det D_x G| dx \\ &\leq \int_{G[\Omega]} f \circ G \circ G^{-1} |\det D_{G^{-1}(x)} G| |\det D_x G^{-1}| dx \\ &= \int_{G[\Omega]} f(x) dx.\end{aligned}$$

- This establishes Theorem (a) for all $f \geq 0$.
- The case $f \in L^1(\mathbb{R}^d)$ follows immediately.
- Theorem (b) is a special case of Theorem (a) where $f = \mathbb{1}_{G[E]}$.
- The case if f is Lebesgue measurable can be handled in much the same ways as the previous theorem, we omit the details. $\square$

Example

Example

We show that

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

Consider

$$\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-x^2-y^2} dxdy$$

and a mapping $G : (0, \infty) \times [0, 2\pi) \to \mathbb{R}^2 \setminus [0, +\infty)$ given by

$$G(r, \theta) = (r\cos(\theta), r\sin(\theta))$$

Then we have

$$D_{r,\theta} G = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}, \quad \det D_{(r,\theta)} = r\cos(\theta)^2 + r\sin(\theta)^2 = r.$$

Example

Thus

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2}dxdy &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r d\theta dr \\
&= 2\pi \int_0^{\infty} re^{-r^2} dr \\
&= \pi \int_0^{\infty} 2re^{-r^2} dr \\
&= \pi \int_0^{\infty} e^{-t} dt = \pi.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}} e^{-x^2} dx = \left(\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-(x^2+y^2)} dxdy\right)^{1/2} = \sqrt{\pi}. \qquad \square$$

Integration in polar coordinates

- The most important nonlinear coordinate system in $\mathbb{R}^2$ and $\mathbb{R}^3$ are **polar coordinates**

$$(r,\theta) \longmapsto (r\cos(\theta), r\sin(\theta)),$$
$$(r,\phi,\theta) \longmapsto (r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta)).$$

- The multi-dimensional change of variables formula yields

$$dx\,dy = r\,dr\,d\theta,$$
$$dx\,dy\,dz = r^2\sin(\phi)\,dr\,d\theta\,d\phi.$$

- Similar coordinate systems exist in higher dimensions, but they become more complicated as the dimension increases.

Polar coordinates

- For most purposes, however, it is sufficient to know that Lebesgue measure is effectively the product of the measure $r^{n-1}dr$ on $\mathbb{R}_+ = (0, \infty)$ and a certain "surface measure σ" on the unit sphere $(d\sigma(\theta) = d\theta$ for $d = 2$; and $d\sigma(\theta, \phi) = \sin(\phi)\, d\theta\, d\phi$ for $d = 3)$.
- Our construction of this measure is motivated by a familiar fact from the plane geometry. Namely, if S_θ is a sector of a disc of radius $r > 0$ with an angle θ, the area $\lambda_2(S_\theta)$ is proportional to θ. In fact, we have

$$\lambda_2(S_\theta) = \frac{1}{2} r^2 \theta.$$

- The equation above can be solved for θ and used to define the angular measure θ in terms of the area of $\lambda_2(S_\theta)$. The same works in $\mathbb{R}^d$ for $d \geq 2$.

Our construction

- Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ be the unit sphere in $\mathbb{R}^d$. If $x \in \mathbb{R}^d \setminus \{0\}$ the **polar coordinates** of x are

$$r = |x| \in (0, \infty) \quad \text{and} \quad x' = \frac{x}{|x|} \in \mathbb{S}^{d-1}.$$

- The map $\Phi(x) = (r, x')$ is a continuous bijection from $\mathbb{R}^d \setminus \{0\}$ to $(0, \infty) \times \mathbb{S}^{d-1}$, whose continuous inverse is

$$\Phi^{-1}(r, x') = rx'.$$

- We denote by m_d the Borel measure on $(0, \infty) \times \mathbb{S}^{d-1}$ induced by Φ from Lebesgue measure on $\mathbb{R}^d$, that is

$$m_d(E) = \lambda_d(\Phi^{-1}[E]).$$

- Moreover, we define the measure $\rho = \rho_d$ on $(0, \infty)$ by

$$\rho(E) = \int_E r^{d-1} dr.$$

Surface measure

Theorem

There is a unique Borel measure $\sigma = \sigma_{d-1}$ on $\mathbb{S}^{d-1}$ such that $\lambda_d = \rho \times \sigma$. If f is Borel measurable on $\mathbb{R}^d$ and $f \geq 0$ or $f \in L^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(rx')r^{d-1}d\sigma(x')dr. \qquad (*)$$

Remark

This theorem can be extended to Lebesgue measurable functions by considering the completion of the measure σ.

Proof. Equation (*) when f is a characteristic function of a set is a restatement of the equation $m_d = \rho \times \sigma$ and it follows for general f by the usual linearity and approximation arguments. Hence we need only to construct σ.

Proof

- If E is a Borel set in $\mathbb{S}^{d-1}$ for $a > 0$ let
$$E_a = \Phi^{-1}((0,a] \times E) = \{rx' : (r,x') \in (0,a] \times E\}.$$
If (*) is to hold with $f = \mathbb{1}_E$, we must have
$$\lambda_d(E_1) = \int_0^1 \int_E r^{d-1} d\sigma(x') dr = \sigma(E) \int_0^1 r^{d-1} dr = \frac{\sigma(E)}{d}.$$
- We therefore define

Definition (Surface measure on $\mathbb{S}^{d-1}$)

$$\sigma(E) = d \cdot \lambda_d(E_1).$$

- Since the map $E \mapsto E_1$ takes Borel sets to Borel sets and commutes with unions, intersections, and complements, it is clear that σ is a Borel measure on $\mathbb{S}^{d-1}$. Also, since E_a is the image of E_1 under the map $x \mapsto ax$, it follows that $\lambda_d(E_a) = a^d \lambda_d(E_1)$.

Proof

- Fix $E \in \mathcal{B}(\mathbb{S}^{d-1})$ and let $\mathcal{A}_E$ be a the collection of finite disjoint unions of sets of the form $(a, b] \times E$. We know that $\mathcal{A}_E$ is an algebra on $\mathbb{R}_+ \times E$ that generates the σ-algebra $\mathcal{M}_E = \{A \times E : A \in \mathcal{B}(\mathbb{R}_+)\}$.
- We show that $m_d = \rho \times \sigma$ on $\mathcal{A}_E$. If $0 < a < b$, then we have

$$\begin{aligned} m_d((a, b] \times E) = \lambda_d(E_b \setminus E_a) = \frac{b^d - a^d}{d}\sigma(E) \\ = \sigma(E) \int_a^b r^{d-1} dr = \rho \times \sigma((a, b] \times E). \end{aligned}$$

- Since $m_d = \rho \times \sigma$ on $\mathcal{A}_E$, hence by the uniqueness assertion of Caratheodory's extension theorem we have $m_d = \rho \times \sigma$ on $\mathcal{M}_E$.
- But $\bigcup_{E \in \mathcal{B}(\mathbb{S}^{d-1})} \mathcal{M}_E$ is precisely the set of Borel rectangles in $\mathbb{R}_+ \times \mathbb{S}^{d-1}$ so another application of the uniqueness theorem shows that $m_d = \rho \times \sigma$ on all Borel sets. $\square$

Remark and Corollary

Corollary

If $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function nonnegative or integrable, such that $f(x) = g(|x|)$ for some measurable function $g : \mathbb{R}_+ \to \mathbb{C}$, then

$$\int_{\mathbb{R}^d} f(x)dx = \sigma(\mathbb{S}^{d-1}) \int_0^\infty g(r)r^{d-1}dr.$$

Proof. By the polar decomposition

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)dx &= \int_0^\infty \int_{\mathbb{S}^{d-1}} f(rx')d\sigma(x')dr \\ &= \int_0^\infty \int_{\mathbb{S}^{d-1}} g(r|x'|)r^{d-1}d\sigma(x')dr \\ &= \sigma(\mathbb{S}^{d-1}) \int_0^\infty g(r)r^{d-1}dr.\end{aligned}$$

$\square$

Proposition

Proposition

For $a > 0$ we have

$$I_d = \int_{\mathbb{R}^d} e^{-a|x|^2} dx = \left(\frac{\pi}{a}\right)^{d/2}.$$

Proof. For $d = 2$ we obtain

$$I_2 = 2\pi \int_0^\infty r e^{-ar^2} dr = \frac{\pi}{a},$$

since $\sigma(\mathbb{S}^1) = 2\pi$. Observe now that $e^{-a|x|^2} = \prod_{j=1}^d e^{-ax_j^2}$. Thus, by Tonelli's theorem, $I_d = (I_1)^d$. In particular, $I_1 = I_2^{1/2}$, so

$$I_d = (I_2)^{d/2} = \left(\frac{\pi}{a}\right)^{d/2}. \qquad \square$$

Surface area of $\mathbb{S}^{d-1}$

Proposition

Recall that $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. *Then one has*

$$\sigma(\mathbb{S}^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Proof. By the previous proposition we have

$$\begin{aligned}\pi^{d/2} &= \int_{\mathbb{R}^d} e^{-|x|^2} dx = \sigma(\mathbb{S}^{d-1}) \int_0^\infty e^{-r^2} r^{d-1} dr \\ &= \frac{1}{2}\sigma(\mathbb{S}^{d-1}) \int_0^\infty s^{d/2-1} e^{-s} ds = \frac{1}{2}\sigma(\mathbb{S}^{d-1})\Gamma(d/2).\end{aligned}$$

Thus

$$2\pi^{d/2}\Gamma(d/2)^{-1} = \sigma(\mathbb{S}^{d-1}). \qquad \square$$

Volume of the unit ball

Proposition

One has

$$\lambda_d(B(0,1)) = \frac{\pi^{d/2}}{\Gamma(d/2+1)},$$

where $B(0,1) = \{x \in \mathbb{R}^d : |x| < 1\}$.

Proof. Since $x\Gamma(x) = \Gamma(x+1)$ we have

$$\begin{aligned}
\lambda_d(B(0,1)) &= \int_{\mathbb{R}^d} \mathbb{1}_{B(0,1)}(x)dx \\
&= \sigma(\mathbb{S}^{d-1}) \int_0^1 r^{d-1} dr \\
&= \frac{1}{d}\sigma(\mathbb{S}^{d-1}) \\
&= \frac{2\pi^{d/2}}{d\Gamma(d/2)} = \frac{\pi^{d/2}}{\Gamma(d/2+1)}.
\end{aligned}$$

□