

Lecture 13

Modes of convergence

A first glance at Banach spaces and L^p spaces

Graduate Real Analysis

Fall Semester

Modes of convergence

Modes of convergence

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of complex-valued functions on a space X .

- ① $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on X iff $\sup_{x \in X} |f_n(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0$. In other words, for any $\varepsilon > 0$ there is $N_\varepsilon \in \mathbb{N}$ such that

$$n \geq N_\varepsilon \quad \text{implies} \quad |f_n(x) - f(x)| < \varepsilon \quad \text{for all} \quad x \in X.$$

- ② $f_n \xrightarrow{n \rightarrow \infty} f$ pointwise iff $|f(x) - f_n(x)| \xrightarrow{n \rightarrow \infty} 0$ for any $x \in X$. In other words, for any $x \in X$ and $\varepsilon > 0$ there is $N_{\varepsilon, x} \in \mathbb{N}$ such that

$$n \geq N_{\varepsilon, x} \quad \text{implies} \quad |f_n(x) - f(x)| < \varepsilon.$$

- ③ Uniform convergence implies pointwise convergence.
- ④ Pointwise convergence implies pointwise convergence a.e.

Convergence in measure

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $L^0(X)$ be the space of all complex-valued measurable functions on X .

Definition (Convergence in measure)

We say that a sequence of measurable functions $(f_n)_{n \in \mathbb{N}} \subseteq L^0(X)$ **converges in measure** to a measurable function $f \in L^0(X)$ if for every $\varepsilon > 0$ we have

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) \xrightarrow{n \rightarrow \infty} 0.$$

Cauchy sequence

We say that a sequence of measurable functions $(f_n)_{n \in \mathbb{N}} \subseteq L^0(X)$ is **Cauchy in measure** if for every $\varepsilon > 0$ we have

$$\mu(\{x \in X : |f_m(x) - f_n(x)| \geq \varepsilon\}) \xrightarrow{m, n \rightarrow \infty} 0.$$

L^1 convergence implies convergence in measure

Proposition

Let $(X, \mathcal{M}, \mu)$ be a measure space, $(f_n)_{n \in \mathbb{N}} \subseteq L^1(X)$ and $f \in L^1(X)$. If $f_n \xrightarrow{n \rightarrow \infty} f$ in $L^1(X)$, which means that $\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0$, then $f_n \xrightarrow{n \rightarrow \infty} f$ in measure.

Proof. For any $\lambda > 0$, recall Chebyshev's inequality

$$\mu(\{x \in X : |h(x)| \geq \lambda\}) \leq \frac{1}{\lambda} \int_E |h(x)| d\mu(x),$$

where $E = \{x \in X : |h(x)| \geq \lambda\}$. Let $E_{n,\varepsilon} = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$. Then by Chebyshev's inequality we have

$$\varepsilon \mu(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon} \int_{E_{n,\varepsilon}} |f_n - f| d\mu \leq \frac{1}{\varepsilon} \int_X |f_n - f| d\mu \xrightarrow{n \rightarrow \infty} 0. \quad \square$$

Remarks

Remarks

- Pointwise convergence does not imply L^1 convergence, it suffices to consider $f_n(x) = n^{-1}\mathbb{1}_{(0,n)}(x)$, which converges to 0 uniformly, but L^1 convergence fails, since $\int_{\mathbb{R}} |f_n(x)| dx = 1$.
- L^1 convergence does not imply pointwise convergence. To see this consider $f_n(x) = \mathbb{1}_{[j2^{-k}, (j+1)2^{-k}]}$ for $n = 2^k + j$ with $0 \leq j < 2^k$. Then we have $\int_0^1 |f_n(x)| dx = 2^{-k}$ for any $2^k \leq n < 2^{k+1}$, but f_n does not converge for any $x \in [0, 1]$, since there are infinitely many $n \in \mathbb{N}$ for which $f_n(x) = 0$ and infinitely many for which $f_n(x) = 1$.
- On the other hand, if $f_n \xrightarrow{n \rightarrow \infty} f$ a.e. on X and $|f_n| \leq g \in L^1(X)$ for all $n \in \mathbb{N}$ then $f_n \xrightarrow{n \rightarrow \infty} f$ in $L^1(X)$, which means $\int_X |f_n - f| d\mu \xrightarrow{n \rightarrow \infty} 0$.
- If $f_n \xrightarrow{n \rightarrow \infty} f$ in $L^1(X)$ then there exists $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $\lim_{k \rightarrow \infty} f_{n_k} = f$ a.e. on X . This will follow from Riesz's theorem.

Riesz theorem

Theorem (Riesz theorem)

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(f_n)_{n \in \mathbb{N}} \subseteq L^0(X)$ be a Cauchy sequence in measure. Then there is a subsequence $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and a measurable function $f \in L^0(X)$ such that

- $f_{n_k} \xrightarrow[k \rightarrow \infty]{} f$ a.e. on X .
- $f_n \xrightarrow[n \rightarrow \infty]{} f$ in measure.
- Moreover, if $f_n \xrightarrow[n \rightarrow \infty]{} g$ in measure, then $f = g$ a.e. on X .

In particular, if $f_n \xrightarrow[n \rightarrow \infty]{} f$ in $L^1(X)$ then there is $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ such that $f_{n_k} \xrightarrow[k \rightarrow \infty]{} f$ a.e. on X .

Proof. $(f_n)_{n \in \mathbb{N}}$ is Cauchy in measure if for every $\varepsilon > 0$, we have

$$\mu(\{x \in X : |f_m(x) - f_n(x)| \geq \varepsilon\}) \xrightarrow[m, n \rightarrow \infty]{} 0.$$

Proof

- We choose a sequence $(g_j)_{j \in \mathbb{N}} = (f_{n_j})_{j \in \mathbb{N}}$ such that

$$\text{if } E_j = \{x \in X : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}, \quad \text{then } \mu(E_j) \leq 2^{-j}.$$

- If $F_k = \bigcup_{j \geq k} E_j$, then

$$\mu(F_k) \leq \sum_{j=k}^{\infty} \mu(E_j) = \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k} \xrightarrow{k \rightarrow \infty} 0.$$

- If $x \notin F_k$, then for any $n \geq m \geq k$ one has

$$|g_m(x) - g_n(x)| \leq \sum_{j=m}^{n-1} |g_{j+1}(x) - g_j(x)| \leq \sum_{j=m}^{n-1} 2^{-j} \leq 2^{1-m} \xrightarrow{m \rightarrow \infty} 0. \quad (*)$$

- We have proved that $(g_n)_{n \in \mathbb{N}}$ is Cauchy in F_k^c . Let

$$F = \bigcap_{k=1}^{\infty} F_k = \limsup_{n \rightarrow \infty} E_n.$$

Proof

- Since $\mu(F_k) \leq 2^{1-k}$ we deduce that

$$\mu(F) \leq \mu(F_k) \leq 2^{1-k}, \quad \text{which implies} \quad \mu(F) = 0.$$

- If we set

$$f(x) = \begin{cases} \lim_{n \rightarrow \infty} g_n(x) & \text{if } x \notin F, \\ 0 & \text{if } x \in F, \end{cases}$$

then we see that f is measurable on X and thus $g_n \xrightarrow{n \rightarrow \infty} f$ a.e.

- If $x \notin F_m$ and $m \geq k$, then by (*) we obtain

$$|g_m(x) - f(x)| \leq 2^{1-m}.$$

- Thus we pick $m \in \mathbb{N}$ so that $2^{1-m} < \varepsilon$ and note that

$$\begin{aligned} & \mu(\{x \in X : |g_m(x) - f(x)| \geq \varepsilon\}) \\ & \leq \mu(\{x \in F_m^c : |g_m(x) - f(x)| \geq \varepsilon\}) + \mu(F_m) \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Proof

- We have shown that $g_k = f_{n_k} \xrightarrow[k \rightarrow \infty]{} f$ in measure, but then $f_n \xrightarrow[n \rightarrow \infty]{} f$ in measure, since

$$\begin{aligned} \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\} &\subseteq \\ \{x \in X : |f_n(x) - f_{n_k}(x)| \geq \varepsilon/2\} &\cup \{x \in X : |f_{n_k}(x) - f(x)| \geq \varepsilon/2\}. \end{aligned}$$

- If $f_n \xrightarrow[n \rightarrow \infty]{} g$ in measure, then

$$\begin{aligned} \{x \in X : |f(x) - g(x)| \geq \varepsilon\} &\subseteq \\ \{x \in X : |f(x) - f_n(x)| \geq \varepsilon/2\} &\cup \{x \in X : |f_n(x) - g(x)| \geq \varepsilon/2\}. \end{aligned}$$

- Since $\varepsilon > 0$ is arbitrary we deduce that

$$\mu(\{x \in X : |f(x) - g(x)| \geq \varepsilon\}) = 0,$$

showing that $f = g$ a.e. on X . □

Three Littlewood principles

Although the notions of measurable sets and measurable functions represent new tools, we should not overlook their relation to the older concepts they replaced. Littlewood aptly summarized these connections in the form of three principles:

Three Littlewood principles

- (a) Every set is nearly a finite union of intervals.
- (b) Every convergent sequence is nearly uniformly convergent.
- (c) Every function is nearly continuous.

Theorem (The first principle: Approximations by rectangles)

If $E \in \mathcal{L}(\mathbb{R}^d)$ and $\lambda_d(E) < \infty$, then for every $\varepsilon > 0$ there is a set A that is a finite union of open rectangles such that $\lambda_d(E \triangle A) < \varepsilon$.

- *The same result is true if μ is a Lebesgue–Stieltjes measure on $\mathbb{R}$.*

The second principle: Egoroff's theorem

Theorem (Egoroff's theorem)

Let $(X, \mathcal{M}, \mu)$ be a finite measure space $\mu(X) < \infty$. Let $(f_n)_{n \in \mathbb{N}} \subseteq L^0(X)$ and $f_n \xrightarrow{n \rightarrow \infty} f \in L^0(X)$ a.e. on X . Then for every $\varepsilon > 0$ there exists a set $E \in \mathcal{M}$ such that $\mu(E) < \varepsilon$ and $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on E^c .

Proof. We may assume (wlog) that $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ for every $x \in X$. Let

$$E_n(k) = \bigcup_{m \geq n} \{x \in X : |f_m(x) - f(x)| \geq k^{-1}\}.$$

For a fixed $k \in \mathbb{N}$, observe that $E_n(k) \supseteq E_{n+1}(k)$ and

$$\bigcap_{n \in \mathbb{N}} E_n(k) = \emptyset.$$

Since $\mu(X) < \infty$ by the continuity of the measure

$$\lim_{n \rightarrow \infty} \mu(E_n(k)) = 0.$$

Proof

Given $\varepsilon > 0$ and $k \in \mathbb{N}$ we choose n_k so large that

$$\mu(E_{n_k}(k)) < \varepsilon 2^{-k}.$$

We now consider

$$E = \bigcup_{k \in \mathbb{N}} E_{n_k}(k).$$

Then

$$\mu(E) \leq \sum_{k \in \mathbb{N}} \mu(E_{n_k}(k)) \leq \sum_{k \in \mathbb{N}} \varepsilon 2^{-k} = \varepsilon,$$

and for $x \in E^c$ we have that

$$|f_n(x) - f(x)| < k^{-1} \quad \text{for } n \geq n_k.$$

Thus $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on E^c . □

Approximations by step functions

Definition (Step functions)

A **step function** on $\mathbb{R}^d$ is a finite linear combination, with complex coefficients, of characteristic functions of rectangles in $\mathbb{R}^d$, i.e.

$$\varphi(x) = \sum_{j=1}^n a_j \mathbb{1}_{R_j},$$

where $R_j \subseteq \mathbb{R}^d$ is a rectangle and $a_j \in \mathbb{C}$. A family of step functions on $\mathbb{R}^d$ is a subfamily of simple functions on $\mathbb{R}^d$.

Theorem (Approximations by step functions)

Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a measurable function. Then there exists a sequence of step functions $(\varphi_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \rightarrow \infty} \varphi_n(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}^d.$$

Proof

- We know that if $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is measurable then there is a sequence $(f_n)_{n \in \mathbb{N}}$ of simple functions such that $0 \leq |f_1| \leq |f_2| \leq \dots \leq |f|$ and $f_n \xrightarrow{n \rightarrow \infty} f$ pointwise everywhere on $\mathbb{R}^d$.
- Thus it suffices to show that if E is a measurable set with finite measure, then $f = \mathbb{1}_E$ can be approximated by step functions.
- We now use the first Littlewood's principle, asserting that for every $\varepsilon > 0$ there are rectangles $Q_1, \dots, Q_k$ so that $\lambda_d(E \triangle \bigcup_{j=1}^k Q_j) < \varepsilon$.
- By considering the grid formed by extending the sides of these cubes, we find almost disjoint rectangles $P_1, \dots, P_m$ such that

$$\bigcup_{j=1}^k Q_j = \bigcup_{j=1}^m P_j \quad \text{and} \quad \lambda_d(E \triangle \bigcup_{j=1}^m P_j) < \varepsilon.$$

- By taking rectangles R_j contained in P_j , and slightly smaller in size, we find a collection of disjoint rectangles that satisfy

$$\lambda_d(E \triangle \bigcup_{j=1}^m R_j) < 2\varepsilon.$$

Proof

- Therefore

$$f(x) = \mathbb{1}_E(x) = \sum_{j=1}^m \mathbb{1}_{R_j}(x)$$

except possibly on a set of measure $< 2\varepsilon$.

- The same remains true if f is a simple function.
- Consequently, for every $n \in \mathbb{N}$, there exists a step function φ_n such that if

$$E_n = \{x \in \mathbb{R}^d : f(x) \neq \varphi_n(x)\},$$

then $\lambda_d(E_n) < 2^{-n}$.

- If we let $F_m = \bigcup_{j=m+1}^{\infty} E_j$ and $F = \bigcap_{m \in \mathbb{N}} F_m$, then $\lambda_d(F) = 0$, since $\lambda_d(F_m) < 2^{-m}$ and

$$\lim_{n \rightarrow \infty} \varphi_n(x) = f(x)$$

on the complement of F , which gives the desired result. □

The third principle: Luzin's theorem

Theorem (Luzin's theorem)

Given a measurable set $E \subseteq \mathbb{R}^d$ and given $f : E \rightarrow \mathbb{C}$, the following statements are equivalent:

- (a) *f is measurable,*
- (b) *for every $\varepsilon > 0$ there exists a closed set $F \subseteq E$ such that $\lambda_d(E \setminus F) < \varepsilon$ and f restricted to the set F is continuous.*

Proof of (a) $\implies$ (b). Assume that $\lambda_d(E) < \infty$ and fix $\varepsilon > 0$.

- Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of step functions such that $f_n \xrightarrow{n \rightarrow \infty} f$ a.e. on $\mathbb{R}^d$. Then we find measurable sets E_n so that $\lambda_d(E_n) < \varepsilon 2^{-n-1}$ and f_n is continuous outside E_n .
- By Egoroff's theorem we find a measurable set U such that $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on U and $\lambda_d(E \setminus U) < \varepsilon/4$.

Proof

- Define a new set

$$D = U \setminus \bigcup_{n \in \mathbb{N}} E_n,$$

then we see that $f_n \xrightarrow{n \rightarrow \infty} f$ uniformly on D and $(f_n)_{n \in \mathbb{N}}$ is a sequence of continuous functions on D , thus f is a continuous function on D .

- Now we find a closed set $F \subseteq D$ such that $\lambda_d(D \setminus F) < \varepsilon/4$. Then it is easy to see that $\lambda_d(E \setminus F) < \varepsilon$.
- If $\lambda_d(E) = \infty$ then we may apply the previous result to the sets $A_k = E \cap \{x \in \mathbb{R}^d : k \leq |x| < k+1\}$ and then use regularity of λ_d .

Proof of (b) $\implies$ (a). We may assume that f is real-valued. Then for $n \in \mathbb{N}$ we can find a closed set $F_n \subseteq E$ such that f is continuous on F_n and $\lambda_d(E \setminus F_n) < n^{-1}$. Define $F = \bigcup_{n \in \mathbb{N}} F_n$ and note that $\lambda_d(E \setminus F) = 0$. We see that for every $a > 0$ the following set

$$\{x \in E : f(x) > a\} = \bigcup_{n \in \mathbb{N}} \{x \in F_n : f(x) > a\} \cup \{E \setminus F : f(x) > a\}$$

is measurable since λ_d is complete. □

Seminorm

Let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$ and let X be a vector space over $\mathbb{K}$. We denote the zero element of X simply by 0 .

- ① By a subspace we shall always mean a vector subspace.
- ② If $x \in X$ we denote by $\mathbb{K}x = \{\lambda x : \lambda \in \mathbb{K}\}$ the one-dimensional space spanned by x .

Seminorm

A **seminorm** on X is a function $X \ni x \mapsto \|x\| \in [0, \infty)$ such that

- ① $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,
- ② $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\lambda \in \mathbb{K}$.

Norm

A seminorm such that $\|x\|_X = 0$ only when $x = 0$ is called a **norm**. A vector space equipped with a norm is called a **normed vector space**.

Norms and metrics and Banach spaces

Norms and metrics

If $(X, \|\cdot\|)$ is a normed vector space the function $\rho(x, y) = \|x - y\|$ is a metric on X . The topology it defines is called the **norm topology on X** .

Banach space

A normed vector space $(X, \|\cdot\|)$ that is complete (Cauchy complete) with respect to the norm metric is called a **Banach space**.

Remark

Every normed vector space can be embedded in a Banach space as a dense space. One way to do this is to mimic the construction of $\mathbb{R}$ from $\mathbb{Q}$ via Cauchy sequences.

Equivalent norms

Equivalent norms

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are **equivalent** if there exists $C \geq 1$ such that

$$C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \text{for any } x \in X.$$

- Equivalent norms define equivalent metrics and hence the same topology and the same Cauchy sequences.

Convergent and absolutely convergent series

If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X , the series $\sum_{n=1}^{\infty} x_n$ is said to **converge** to $x \in X$ if

$$\sum_{n=1}^N x_n \xrightarrow{N \rightarrow \infty} x \quad \Longleftrightarrow \quad \lim_{N \rightarrow \infty} \left\| \sum_{n=1}^N x_n - x \right\| = 0,$$

and it is called **absolutely convergent** if $\sum_{n \in \mathbb{N}} \|x_n\| < \infty$.

Completeness criterion in normed spaces

Theorem

A normed vector space $(X, \|\cdot\|)$ is complete iff every absolutely convergent series in X converges.

Proof ($\Rightarrow$). Assume that X is complete and $\sum_{n \in \mathbb{N}} \|x_n\| < \infty$. Define

$$S_n = \sum_{m=1}^n x_m \quad \text{for every } n \in \mathbb{N}.$$

Then for $n > k$ we have

$$\|S_n - S_k\| = \left\| \sum_{m=k+1}^n x_m \right\| \leq \sum_{m=k+1}^n \|x_m\| \leq \sum_{m=k+1}^{\infty} \|x_m\| \xrightarrow[k \rightarrow \infty]{} 0.$$

So $(S_n)_{n \in \mathbb{N}}$ converges since X is complete.

Proof

Proof ($\Leftarrow$). Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in X . For any $k \in \mathbb{N}$ there is $n_k \in \mathbb{N}$ such that for every $n, m \geq n_k$ we have

$$\|x_n - x_m\| \leq 2^{-k}.$$

Thus we find $n_1 < n_2 < \dots$ such that for every $k \in \mathbb{N}$ one has

$$\|x_{n_{k+1}} - x_{n_k}\| \leq 2^{-k}.$$

Define $y_0 = x_{n_1}$ and $y_k = x_{n_{k+1}} - x_{n_k}$ for $k \geq 1$. Then the sequence

$$\sum_{k \in \mathbb{N}} \|y_k\| = \|x_{n_1}\| + \sum_{k \in \mathbb{N}} \|x_{n_{k+1}} - x_{n_k}\| \leq \|x_{n_1}\| + \sum_{k \in \mathbb{N}} 2^{-k} < \infty,$$

thus $\sum_{k \in \mathbb{N}} y_k$ converges.

Proof

But

$$x_{n_k} = \sum_{j=0}^{k-1} y_j = \sum_{j=0}^{k-1} (x_{n_{j+1}} - x_{n_j}),$$

hence the sequence $(x_{n_k})_{k \in \mathbb{N}}$ converges as well. Let $\lim_{k \rightarrow \infty} x_{n_k} = x$.

We show that

$$\lim_{n \rightarrow \infty} x_n = x.$$

Indeed, $\|x_n - x_m\| < \frac{\varepsilon}{2}$ for all $n, m \geq N_1$, and $\|x_{n_k} - x\| < \frac{\varepsilon}{2}$ for all $k \geq N_2$.

Let $n, k \geq N = \max\{N_1, N_2\}$, then $n_k \geq k \geq N_2$ and $n \geq N_1$ yielding

$$\|x_n - x\| \leq \|x_n - x_{n_k}\| + \|x_{n_k} - x\| < \varepsilon.$$



Examples

Bounded and bounded continuous functions are Banach spaces

If X is a topological space then

$$B(X) = \{f : X \rightarrow \mathbb{C} : \sup_{x \in X} |f(x)| = \|f\|_{\infty} < \infty\}$$

and

$$BC(X) = \{f : X \rightarrow \mathbb{C} : f \text{ is continuous and } \sup_{x \in X} |f(x)| = \|f\|_{\infty} < \infty\}$$

are Banach spaces with the uniform norm

$$\|f\|_{\infty} = \sup_{x \in X} |f(x)|.$$

It is straightforward to show that $f \mapsto \|f\|_{\infty}$ defines a norm on $B(X)$ and $BC(X)$ and both spaces are complete with respect to this norm.

L^p spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $0 < p < \infty$ and a measurable function $f : X \rightarrow \mathbb{C}$ we define the quantity

$$\|f\|_{L^p} = \|f\|_{L^p(X)} = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p},$$

and for $p = \infty$ we set

$$\|f\|_{L^\infty} = \|f\|_{L^\infty(X)} = \operatorname{esssup}_{x \in X} |f(x)|,$$

where

$$\operatorname{esssup}_{x \in X} |f(x)| = \inf \{ C \geq 0 : \mu(\{x \in X : |f(x)| \geq C\}) = 0 \}.$$

Definition (L^p spaces)

For every $0 < p \leq \infty$ we define

$$L^p(X, \mathcal{M}, \mu) = \{f : X \rightarrow \mathbb{C} : f \text{ is measurable and } \|f\|_{L^p(X)} < \infty\}.$$

Basic remarks on L^p spaces

Remarks

- We abbreviate $L^p(X, \mathcal{M}, \mu)$ by $L^p(X, \mu)$ or $L^p(X)$ or simply L^p when this will cause no confusions.
- As we have done with $L^1(X)$, we consider two functions to define the same element of $L^p(X)$ when they are equal almost everywhere.
- $L^p(X)$ is a vector space. For $p = \infty$ there is nothing to prove. For $0 < p < \infty$ it follows from a simple inequality

$$|f + g|^p \leq (2 \max\{|f|, |g|\})^p \leq 2^p(|f|^p + |g|^p) \quad \text{for } f, g \in L^p(X).$$

- If X is a countable set we define $\ell^p(X)$ to be $L^p(X, \mathcal{P}(X), \mu)$, where μ is counting measure on $(X, \mathcal{P}(X))$, then

$$\|f\|_{\ell^p(X)} = \left(\sum_{x \in X} |f(x)|^p \right)^{1/p} \quad \text{for } 0 < p < \infty.$$

Hölder's inequality

Proposition (Hölder's inequality)

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $f \in L^p(X)$ and $g \in L^q(X)$, we have

$$\|fg\|_{L^1(X)} \leq \|f\|_{L^p(X)} \|g\|_{L^q(X)}.$$

The parameter $p' = q = \frac{p}{p-1}$ is called the conjugate exponent to p . We will use convention that $1' = \infty$ and $\infty' = 1$.

Proof. By homogeneity, we may assume that $\|f\|_{L^p(X)} = \|g\|_{L^q(X)} = 1$. Note that

$$\begin{aligned} \|fg\|_{L^1(X)} &= \int_X |fg| d\mu = \int_X (|f|^p)^{1/p} (|g|^q)^{1/q} d\mu \leq \int_X \max(|f|^p, |g|^q) d\mu \\ &\leq \int_X |f|^p d\mu + \int_X |g|^q d\mu = 2 = 2\|f\|_{L^p(X)} \|g\|_{L^q(X)}. \end{aligned}$$

Proof.

- Thus, we have proved that $\|fg\|_{L^1(X)} \leq 2\|f\|_{L^p(X)}\|g\|_{L^q(X)}$. That is, Hölder's inequality holds, but with a constant of 2 instead of 1.
- We use **the best constant argument** to complete the proof. Let $C > 0$ be the smallest constant such that, for all measure spaces $(X, \mathcal{M}, \mu)$, we have

$$\|fg\|_{L^1(X)} \leq C\|f\|_{L^p(X)}\|g\|_{L^q(X)}.$$

- We know that $C \leq 2$, and we will show that $C \leq 1$ using the “**tensor trick**.” Let $F(x, y) = f(x)f(y)$ and $G(x, y) = g(x)g(y)$. Then

$$\begin{aligned}\|fg\|_{L^1(X)}^2 &= \|FG\|_{L^1(X \times X)} \leq C\|F\|_{L^p(X \times X)}\|G\|_{L^q(X \times X)} \\ &= C\|f\|_{L^p(X)}^2\|g\|_{L^q(X)}^2.\end{aligned}$$

- Taking square roots, we have $\|fg\|_{L^1(X)} \leq C^{1/2}\|f\|_{L^p(X)}\|g\|_{L^q(X)}$, so $C \leq C^{1/2}$, (since C is the best constant), which implies $C \leq 1$. $\square$

Minkowski's inequality

Proposition (Minkowski's inequality)

Let $(X, \mathcal{M}, \mu)$ be a measure space and let $1 \leq p \leq \infty$. Then, for any $f, g \in L^p(X)$, we have

$$\|f + g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}.$$

Proof. For $f, g \in L^p(X)$ note that

$$|f + g|^p \leq |f| |f + g|^{p-1} + |g| |f + g|^{p-1}.$$

Then by Hölder's inequality, with $q = \frac{p}{p-1}$, we immediately obtain

$$\begin{aligned} \|f + g\|_{L^p(X)}^p &\leq (\|f\|_{L^p(X)} + \|g\|_{L^p(X)}) \| |f + g|^{p-1} \|_{L^q(X)} \\ &\leq (\|f\|_{L^p(X)} + \|g\|_{L^p(X)}) \|f + g\|_{L^p(X)}^{p/q}. \end{aligned}$$

Dividing both side by $\|f + g\|_{L^p(X)}^{p/q} = \|f + g\|_{L^p(X)}^{p-1}$ we obtain Minkowski's inequality. □

L^p spaces are normed vector spaces

Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $1 \leq p \leq \infty$ the set $L^p(X)$ is a norm vector space with the norm $\|\cdot\|_{L^p(X)}$, where

$$\|f\|_{L^p(X)} = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}, \quad \text{for } 1 < p < \infty,$$

and

$$\|f\|_{L^\infty(X)} = \operatorname{esssup}_{x \in X} |f(x)|, \quad \text{for } p = \infty.$$

Proof. Let $f, g \in L^p(X)$ be given.

- It is obvious that $\|f\|_{L^p(X)} = 0$ if and only if $f = 0$ a.e. on X .
- It is also easy to see that $\|\lambda f\|_{L^p(X)} = |\lambda| \|f\|_{L^p(X)}$ for any $\lambda \in \mathbb{C}$.
- By Minkowski's inequality we also have the triangle inequality

$$\|f + g\|_{L^p(X)} \leq \|f\|_{L^p(X)} + \|g\|_{L^p(X)}.$$

□

L^p spaces are Banach spaces

Theorem

Let $(X, \mathcal{M}, \mu)$ be a measure space. For every $1 \leq p \leq \infty$ the set $L^p(X)$ with the norm $\|\cdot\|_{L^p(X)}$ is a Banach space.

Proof. To prove completeness it suffices to show that every absolutely convergent series in $L^p(X)$ converges.

- Suppose that $(f_k)_{k \in \mathbb{N}} \subseteq L^p(X)$ and $\sum_{k \in \mathbb{N}} \|f_k\|_{L^p(X)} = B < \infty$.
- Let $G_n = \sum_{k=1}^n |f_k|$ and $G = \sum_{k \in \mathbb{N}} |f_k|$. Then $G_n \in L^p(X)$, and $G_n \leq G_{n+1}$ and $\|G\|_{L^p(X)}^p = \lim_{n \rightarrow \infty} \|G_n\|_{L^p(X)}^p \leq B^p$ by the **(MCT)**.
- Hence $G \in L^p(X)$ and $G(x) < \infty$ a.e. on X , which ensures that $F = \sum_{k \in \mathbb{N}} f_k < \infty$ a.e. on X . Since $|F| \leq G$ we obtain $F \in L^p(X)$.
- Then $|F - \sum_{k=1}^n f_k|^p \leq (2G)^p \in L^1(X)$, so by the **(DCT)** we have

$$\left\| F - \sum_{k=1}^n f_k \right\|_{L^p(X)} \xrightarrow{n \rightarrow \infty} 0.$$

- Thus the series $\sum_{k=1}^n f_k$ converges in the $L^p(X)$ norm as desired. $\square$