Lecture 14

More about Banach spaces, Hahn–Banach theorems and extensions of Lebesgue measure

Graduate Real Analysis

Fall Semester

Bounded linear mapping

Definition (Bounded linear mapping)

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. A map $T : X \to Y$ between two normed vector spaces is **linear** if it satisfies

(i) **(Homogenity)**

$$T(\lambda x) = \lambda T(x), \quad \text{for} \quad x \in X, \ \lambda \in \mathbb{C}.$$

(ii) **(Additivity)**

$$T(x_1 + x_2) = T(x_1) + T(x_2), \quad \text{for} \quad x_1, x_2 \in X.$$

Moreover, a linear map $T : X \to Y$ between two normed vector spaces is **bounded** if there exists $C \geq 0$ such that

$$\|Tx\|_Y \leq C\|x\|_X \quad \text{for all} \quad x \in X.$$

Continuity of linear maps

Theorem

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed vector spaces and $T : X \to Y$ be a linear map. Then the following are equivalent:

(i) *T is continuous at a point,*

(ii) *T is continuous at every point,*

(iii) *T is bounded.*

Proof. The implication (ii) $\implies$ (i) is obvious, so it suffices to establish the implications (iii) $\implies$ (ii) and (i) $\implies$ (iii).

Proof (iii) $\implies$ (ii). Let $x \in X$ and $x_n \xrightarrow[n\to\infty]{} x$ in X, then by linearity and boundedness we obtain

$$\|Tx_n - Tx\|_Y = \|T(x_n - x)\|_Y \leq C\|x_n - x\|_X \xrightarrow[n\to\infty]{} 0,$$

so T is continuous at $x \in X$. $\square$

Proof

Proof (i) $\Longrightarrow$ (iii). We first show that T is continuous at 0. Let $x_n \xrightarrow[n\to\infty]{} 0$ in X, then $u_n = x_n + x_0 \xrightarrow[n\to\infty]{} x_0$ implying

$$Tx_n + Tx_0 = T(x_n + x_0) = Tu_n \xrightarrow[n\to\infty]{} Tx_0,$$

so

$$Tx_n \xrightarrow[n\to\infty]{} 0 = T(0).$$

- Suppose for a contradiction that (iii) is not true. This means that for every $n \in \mathbb{N}$ there is $x_n \in X$ such that

$$\|Tx_n\| > n\|x_n\|.$$

- In particular, $x_n \neq 0$. Define $u_n = \frac{1}{\sqrt{n}} \frac{x_n}{\|x_n\|}$, then $\|u_n\| = \frac{1}{\sqrt{n}}$ and $u_n \xrightarrow[n\to\infty]{} 0$. Thus

$$\|Tu_n\| = \frac{1}{\sqrt{n}} \frac{\|Tx_n\|}{\|x_n\|} > \frac{n}{\sqrt{n}} = \sqrt{n} \xrightarrow[n\to\infty]{} \infty,$$

which is impossible, since $\|Tu_n\| \xrightarrow[n\to\infty]{} 0.$ $\square$

Operator norm

Definition (Operator norm of a bounded operator)

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normd vector spaces, we denote the space of all bounded linear maps from X to Y by $L(X, Y)$. It is easily verified that $L(X, Y)$ is a vector space and that the function

$$L(X, Y) \ni T \mapsto \|T\|_{X \to Y} \in [0, \infty)$$

defined by

$$\begin{aligned} \|T\|_{X \to Y} &= \sup\{\|Tx\|_Y : \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|_Y}{\|x\|_X} : x \neq 0\right\} \\ &= \inf\{C \geq 0 : \|Tx\|_Y \leq C\|x\|_X \ \text{ for all } \ x \in X\} \end{aligned}$$

it is a norm on $L(X, Y)$ called the **operator norm**.

Proposition

Proposition

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normd vector spaces. If Y is complete so is $L(X, Y)$.

Proof. Let $(T_n)_{n\in\mathbb{N}} \subseteq L(X, Y)$ be a Cauchy sequence, then

$$\|T_n - T_m\|_{X\to Y} \xrightarrow[m,n\to\infty]{} 0.$$

Thus for every $x \in X$ we have

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\|_{X\to Y}\|x\|_X \xrightarrow[m,n\to\infty]{} 0,$$

hence $(Tx_n)_{n\in\mathbb{N}}$ is Cauchy in Y. So it converges since Y **is complete**. Let

$$Tx = \lim_{n\to\infty} Tx_n \in Y.$$

We have to show that T is bounded and $\|T - T_n\|_{X\to Y} \xrightarrow[n\to\infty]{} 0$.

Proof

- Fix $\varepsilon > 0$ then there is $N_\varepsilon \in \mathbb{N}$ such that for every $n, m \geq N_\varepsilon$ we have $\|T_n - T_m\|_{X\to Y} < \varepsilon$. Let $n \geq N_\varepsilon$, then

$$\|(T_n - T)x\|_Y = \|T_n x - Tx\|_Y = \lim_{m\to\infty} \|T_n x - T_m x\|_Y.$$

- On the other hand, if $m \geq N_\varepsilon$, then

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|_{X\to Y}\|x\|_X \leq \varepsilon\|x\|_X.$$

- Hence, if $n \geq N_\varepsilon$, then

$$\|(T_n - T)x\|_Y \leq \varepsilon\|x\|_X, \quad \text{which gives} \quad \|T_n - T\|_{X\to Y} \leq \varepsilon.$$

- In particular, $T_{N_\varepsilon} - T$ and T_{N_ε} are bounded thus

$$\|T\|_{X\to Y} \leq \|T - T_{N_\varepsilon}\|_{X\to Y} + \|T_{N_\varepsilon}\|_{X\to Y} \leq \varepsilon + \|T_{N_\varepsilon}\|_{X\to Y}.$$

- Hence we proved that $T \in L(X, Y)$ and $\|T_n - T\|_{X\to Y} \xrightarrow[n\to\infty]{} 0$. $\square$

Space of linear functionals X^*

Definition (Linear functional)

Let $(X, \|\cdot\|_X)$ be a vector space over $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A linear map from X to $\mathbb{K}$ is called a **linear functional** on X.

Definition (Dual space)

If $(X, \|\cdot\|_X)$ is a normed space over $\mathbb{K}$, the space $L(X, \mathbb{K})$ of bounded linear functionals on X is called the **dual space** of X and it is denoted by

$$X^* = L(X, \mathbb{K}).$$

Theorem

Let $(X, \|\cdot\|_X)$ be a vector space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then the dual space $(X^*, \|\cdot\|_{X^*})$ is a Banach space with the operator norm, i.e.

$$\|f\|_{X^*} = \|f\|_{X\to\mathbb{K}} \quad \text{for every} \quad f \in X^*.$$

Examples

Example 1

Let K be a compact Hausdorff topological space. Let $(K, \mathrm{Bor}(K), \mu)$ be a Borel measure space. Let $X = C(K)$ be the space of all continuous functions on K. Then

$$\Lambda(f) = \int_K f(x) d\mu(x), \quad \text{for} \quad f \in X$$

defies a bounded linear functional on X.

Example 2

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, and let $g \in L^q(X)$, with $1 \leq q \leq \infty$. Then

$$\Lambda(f) = \int_X f(x)\overline{g(x)} d\mu(x)$$

defines a bounded linear functional on $L^p(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Sublinear functionals

Definition

Suppose X is a vector space. A map $p : X \to \mathbb{R}$ is called a sublinear functional if it satisfies

(a) **(positive homogeneity)** $p(\alpha x) = \alpha p(x)$ for all $\alpha \geq 0$ and $x \in X$,

(b) **(subadditivity)** $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

Condition (a) implies that $p(0) = 0$. Condition (b) is also called the triangle inequality.

Examples

- Any linear functional is a sublinear functional.
- Every seminorm is a sublinear functional.

From sublinear to linear functionals

Proposition

A sublinear functional p on a real vector space X is linear if and only if $p(-x) = -p(x)$ for all $x \in X$.

Proof. Certainly, if p is linear, then the conclusion follows. Suppose now that $p(-x) = -p(x)$ for all $x \in X$. Then, by the triangle inequality,

$$p(x+y) = -p(-x-y) \geq -(p(-x) + p(-y)) = p(x) + p(y).$$

The reverse inequality is simply subadditivity of p. $\square$

Definition (Ordering between sublinear functionals)

Let $\mathfrak{B}_X$ be the collection of all sublinear functionals on a real vector space X. We define an order on the set $\mathfrak{B}_X$ by saying $p \leq q$ whenever

$$p(x) \leq q(x) \quad \text{for all} \quad x \in X.$$

Hahn–Banach extension theorem

Theorem (Hahn–Banach extension theorem)

Let X be a real vector space and let p be a sublinear functional on X. If V is a subspace of X and if $f : V \to \mathbb{R}$ is a linear functional such that

$$f(x) \leq p(x) \quad \text{for} \quad x \in V,$$

then there is an extension $\hat{f} : X \to \mathbb{R}$ of f that is linear and satisfies

$$\hat{f}(x) \leq p(x) \quad \text{for} \quad x \in X,$$

Proof. We begin by showing that if $x \in X \setminus V$, then f can be extended to a linear functional g on $V + \mathbb{R}x$ satisfying $g(y) \leq p(y)$ for $y \in V + \mathbb{R}x$. If $y_1, y_2 \in V$, we have

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(x + y_2).$$

Proof

- Equivalently, for every $y_1, y_2 \in V$ we have that

$$f(y_1) - p(y_1 - x) \leq p(x + y_2) - f(y_2).$$

- Hence

$$\sup\{f(y) - p(y - x) : y \in V\} \leq \inf\{p(x + y) - f(y) : y \in V\}.$$

- Choose $\alpha \in \mathbb{R}$ such that

$$\sup\{f(y) - p(y - x) : y \in V\} \leq \alpha \leq \inf\{p(x + y) - f(y) : y \in V\},$$

  and define $g : V + \mathbb{R}x \to \mathbb{R}$ by setting

$$g(y + \lambda x) = f(y) + \lambda\alpha.$$

Proof

- Then, g is clearly linear, and $g_{|V} = f$, so $g(y) \leq p(y)$ for $y \in V$.
- If $\lambda > 0$ and $y \in V$, then

$$\begin{aligned} g(y + \lambda x) = \lambda[f(y/\lambda) + \alpha] &\leq \lambda[f(y/\lambda) + p(x + (y/\lambda)) - f(y/\lambda)] \\ &= p(y + \lambda x). \end{aligned}$$

- If $\lambda = -\mu < 0$, then

$$\begin{aligned} g(y + \lambda x) = \mu[f(y/\mu) - \alpha] &\leq \mu[f(y/\mu) - f(y/\mu) + p((y/\mu) - x)] \\ &= p(y + \lambda x). \end{aligned}$$

- Thus $g(y) \leq p(y)$ for $y \in V + \mathbb{R}x$.
- Evidently, the same reasoning can be applied to any linear extension F of f satisfying $F \leq p$ on its domain and it shows that the domain of a maximal linear extension satisfying $F \leq p$ must be the whole space X.

Proof

- Let $\mathcal{F}$ be the family of linear extensions (F, M) of (f, V), where M is a linear subspace of X containing V and F is a linear functional on M such that $F(x) \le p(x)$ for all $x \in M$, and $F(x) = f(x)$ for all $x \in V$.
- The family $\mathcal{F}$ is partially ordered by the relation

$$(f_1, M_1) \prec (f_2, M_2) \iff M_1 \subset M_2 \text{ and } f_1 = f_2 \text{ on } M_1.$$

- Since the union of any increasing family of subspaces of X is again a subspace, one easily sees that the union of any linearly ordered subfamily of $\mathcal{F}$ lies in $\mathcal{F}$.
- The proof is therefore completed by invoking Kuratowski–Zorn's lemma, as the maximal element $(\hat{f}, M) \in \mathcal{F}$ is the extension with the desired properties. We must have $M = X$, otherwise we could find an extension of $\hat{f}$, but this would contradict maximality of $(\hat{f}, M)$. $\square$

Invariant Hahn–Banach theorem

Theorem (Invariant Hahn–Banach theorem)

Let X be a real vector space and let p be a sublinear functional on X. Let $f : V \to \mathbb{R}$ be a linear functional on a linear subspace V of X such that $f \leq p$ on V. Suppose that $\mathcal{T}$ is a commutative collection of linear maps on X (i.e. $ST = TS$ for all $S, T \in \mathcal{T}$) such that

(a) $T[V] \subseteq V$ *for any* $T \in \mathcal{T}$,

(b) $p(Tx) \leq p(x)$ *for any* $T \in \mathcal{T}$ *and* $x \in X$,

(c) $f(Tx) = f(x)$ *for any* $T \in \mathcal{T}$ *and* $x \in V$.

Then there is an extension $\hat{f} : X \to \mathbb{R}$ of f that is linear and satisfies

(i) $\hat{f}(x) = f(x)$ *for* $x \in V$.

(ii) $\hat{f}(x) \leq p(x)$ *for any* $x \in X$.

(iii) $\hat{f}(Tx) = \hat{f}(x)$ *for any* $T \in \mathcal{T}$ *and* $x \in X$.

Let $\mathfrak{B}_X$ be the collection of all sublinear functionals on a real vector space X.

Proof

Proof. Consider the collection

$$\mathcal{C} = \{q \in \mathfrak{B}_X : q \leq p \ \text{ and } \ f \leq q \text{ on } V \text{ and } q \circ T \leq q \text{ for all } T \in \mathcal{T}\}.$$

We will use Zorn's Lemma to show there exists a minimal element $q \in \mathcal{C}$.

- Suppose $\mathcal{L} = \{q_i : i \in I\}$ is a chain in $\mathcal{C}$. Note that $q = \inf_{i \in I} q_i$ is a sublinear functional on X. Why? Justify this! Then

$$q(Tx) \leq q_i(Tx) \leq q_i(x), \quad \text{for all} \quad x \in X, \ T \in \mathcal{T}, \ i \in I,$$

  since $q_i \in \mathcal{C}$ for all $i \in I$. Hence $q(Tx) \leq q(x)$ for all $T \in \mathcal{T}$ and $x \in X$ and consequently $q \in \mathcal{C}$ is a lower bound for the chain $\mathcal{L}$.
- Therefore, $\mathcal{C}$ contains a minimal element, by Zorn's Lemma. Now let q be a minimal element of $\mathcal{C}$ and let $T \in \mathcal{T}$, $n \in \mathbb{N}$ and define

$$q_n(x) = q\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j x\right), \quad x \in X.$$

- Note that q_n is sublinear since q is sublinear and T is linear.

Proof

- Moreover, $q_n \in \mathcal{C}$, since for any $S \in \mathcal{T}$ we have

$$q_n(Sx) = q\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j Sx\right) = q\left(S\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j x\right)\right) \leq q_n(x),$$

and $q_n(x) = q\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j x\right) \geq f\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j x\right) = f(x)$ for $x \in V$.

- By sublinearity and the fact that $q(T^{n-1}x) \leq q(x)$ for $x \in X$, we have

$$q_n(x) = q\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j x\right) \leq q\left(\frac{x}{n}\right) + \ldots + q\left(\frac{x}{n}\right) = q(x),$$

and by minimality of q we obtain that $q = q_n$ for all $n \in \mathbb{N}$. Hence

$$q(x) = q\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j x\right), \quad x \in X.$$

Proof

- We have proved that $f(x) \leq q(x) \leq p(x)$ for all $x \in V$. Thus by the Hahn–Banach theorem there exists an extension $\hat{f} : X \to \mathbb{R}$ of f such that
$$\hat{f}(x) \leq q(x) \leq p(x) \quad \text{for} \quad x \in X.$$
- We have to prove that $\hat{f}(Tx) = \hat{f}(x)$ for any $T \in \mathcal{T}$ and $x \in X$. For this purpose note that for any $x \in X$ and $n \in \mathbb{N}$ we have
$$\begin{aligned} q(x - Tx) &= q\left(\frac{1}{n}\sum_{j=0}^{n-1} T^j(x - Tx)\right) = q\left(\frac{1}{n}(x - T^n x)\right) \\ &\leq \frac{1}{n}q(x) + \frac{1}{n}q(-T^n x) \leq \frac{1}{n}(q(x) + q(-x)) \xrightarrow[n\to\infty]{} 0. \end{aligned}$$
- Thus $\hat{f}(x) - \hat{f}(Tx) = \hat{f}(x - Tx) \leq q(x - Tx) \leq 0$, and a similar argument shows that $\hat{f}(Tx) - \hat{f}(x) \leq q(-x - T(-x)) \leq 0$, yielding the desired conclusion. $\square$

Extension of Lebesgue measure to all subsets of $\mathbb{R}$

Theorem

There exits a nonnegative set function μ defined for all bounded subsets of $\mathbb{R}$ such that

(a) $\mu(A \cup B) = \mu(A) + \mu(B)$ *if* $A \cap B = \emptyset$.

(b) $\mu(x + A) = \mu(A)$ *for every* $A \subseteq \mathbb{R}$ *and every* $x \in \mathbb{R}$.

(c) $\mu(A) = \lambda(A)$ *for every bounded* $A \in \mathcal{L}(\mathbb{R})$.

In other words, Lebesgue measure λ on $\mathbb{R}$ can be extended to an additive translation-invariant set function defined on all subsets of $\mathbb{R}$.

Proof. Let $B_{fin}(\mathbb{R})$ be the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ such that $\{x \in \mathbb{R} : f(x) \neq 0\} \subseteq [a, b]$ for some compact interval $[a, b]$.

Let $BL_{fin}(\mathbb{R})$ be the set of all Lebesgue measurable functions in $B_{fin}(\mathbb{R})$.

- Define $p : B_{fin}(\mathbb{R}) \to \mathbb{R}$ by

$$p(f) = \inf\left\{ \int_{\mathbb{R}} g(x)dx : g \in BL_{fin}(\mathbb{R}) \text{ and } |f| \leq g \right\}.$$

Proof

- Note that $p(f)$ is finite. Indeed, if $f \in B_{fin}(\mathbb{R})$ then $|f(x)| \leq m$ for some $m \geq 0$, so we may take $g = m\mathbb{1}_{[a,b]}$. Then $|f(x)| \leq g(x)$ and $p(f) \leq m(b-a) < \infty$.
- It is not difficult to see that p is a sublinear functional.
- Define a translation operator on $BL_{fin}(\mathbb{R})$ by

$$A_t f(x) = f(x+t) \quad \text{for} \quad x, t \in \mathbb{R}.$$

- Then we see that $A_t A_s = A_s A_t$ for all $s, t \in \mathbb{R}$ and $A_0 f = f$ for $f \in BL_{fin}(\mathbb{R})$ and $A_t\big[BL_{fin}(\mathbb{R})\big] \subseteq BL_{fin}(\mathbb{R})$ for $t \in \mathbb{R}$.
- Moreover, $p(A_t f) \leq p(f)$ for any $f \in BL_{fin}(\mathbb{R})$. Indeed, if $|f| \leq g$, then $|A_t f| \leq A_t g$ and

$$p(A_t f) \leq \int_{\mathbb{R}} A_t g(x)dx = \int_{\mathbb{R}} g(x+t)dx = \int_{\mathbb{R}} g(x)dx,$$

giving the claim $p(A_t f) \leq p(f)$.

Proof

- We define a functional
$$\pi(f) = \int_{\mathbb{R}} f(x)dx \quad \text{for} \quad f \in BL_{fin}(\mathbb{R}).$$
- Then we see that $\pi(f) \leq p(f)$, since
$$\pi(f) = \int_{\mathbb{R}} f(x)dx \leq \int_{\mathbb{R}} |f(x)|dx = p(f),$$
and we also have $\pi(A_t f) = \pi(f)$ for any $f \in BL_{fin}(\mathbb{R})$ and $t \in \mathbb{R}$.
- By the invariant Hahn–Banach theorem there exists an extension $\Pi : B_{fin}(\mathbb{R}) \to \mathbb{R}$ of π such that
  (i) $\Pi(f) = \pi(f)$ for all $f \in BL_{fin}(\mathbb{R})$.
  (ii) $\Pi(f) \leq p(f)$ for all $f \in B_{fin}(\mathbb{R})$.
  (iii) $\Pi(A_t f) = \Pi(f)$ for all $f \in B_{fin}(\mathbb{R})$ and $t \in \mathbb{R}$.

Proof

- We show that $\Pi(f) \geq 0$ if $f \geq 0$ and $f \in B_{fin}(\mathbb{R})$. Suppose that $f \leq m\mathbb{1}_{[a,b]}$ for some $m \geq 0$. Then $0 \leq m\mathbb{1}_{[a,b]} - f \leq m\mathbb{1}_{[a,b]}$, and

$$\Pi(m\mathbb{1}_{[a,b]} - f) \leq p(m\mathbb{1}_{[a,b]} - f) \leq \int_{\mathbb{R}} m\mathbb{1}_{[a,b]}(x)dx = m(b-a).$$

- Hence $m(b-a) - \Pi(f) = \pi(m\mathbb{1}_{[a,b]}) - \Pi(f) \leq m(b-a)$ and consequently $\Pi(f) \geq 0$, since $\pi(m\mathbb{1}_{[a,b]}) = m(b-a)$.
- Now for a bounded set $A \subset \mathbb{R}$ we define $\mu(A) = \Pi(\mathbb{1}_A)$. This set functions has all desired properties:
  (a) $\mu(A) \geq 0$ for all bounded $A \subset \mathbb{R}$.
  (b) For any $A, B \subset \mathbb{R}$ such that $A \cap B = \emptyset$ we have

$$\mu(A \cup B) = \Pi(\mathbb{1}_{A \cup B}) = \Pi(\mathbb{1}_A) + \Pi(\mathbb{1}_B) = \mu(A) + \mu(B).$$

  (c) For a bounded set $A \in \mathcal{L}(\mathbb{R})$ we also have

$$\mu(A) = \Pi(\mathbb{1}_A) = \int_{\mathbb{R}} \mathbb{1}_A(x)dx = \lambda(A). \qquad \square$$

Remarks

- By a similar argument Lebesgue measure on $\mathbb{R}^2$ can be extended to a finitely additive measure on $\mathcal{P}(\mathbb{R}^2)$ that is invariant under translations and rotations. The Banach–Tarski paradox prevents this result from being extended to higher dimensions.
- The Banach–Tarski paradox asserts that if $U, V \subset \mathbb{R}^d$ are arbitrary bounded open sets, $d \geq 3$, then there exists $k \in \mathbb{N}$ and subsets $E_1, \ldots, E_k, F_1, \ldots, F_k$ of $\mathbb{R}^d$ such that
  - $E_i \cap E_j = \emptyset$ whenever $i \neq j$, and $\bigcup_{j=1}^{k} E_j = U$;
  - $F_i \cap F_j = \emptyset$ whenever $i \neq j$, and $\bigcup_{j=1}^{k} F_j = V$;
  - $E_j \sim F_j$ for all $j = 1, \ldots, k$.

  Here $A \sim B$ means that A can be mapped into B by a combination of a translation, a rotation, and a reflection.
- Thus one can cut up a ball the size of a pea into a finite number of pieces and rearrange them to form a ball the size of the earth! Needless to say, the sets E_j and F_j are very bizarre.

Simple fact about linear functionals

Proposition

Let X be a vector space over $\mathbb{C}$. If $f : X \to \mathbb{C}$ is a linear functional and $u = \mathrm{Re} f$, then u is a real linear functional, and $f(x) = u(x) - iu(ix)$ for all $x \in X$. Conversely, if u is a real linear functional on X and $f : X \to \mathbb{C}$ is defined by $f(x) = u(x) - iu(ix)$, then f is complex linear. In this case, if X is normed, we have $\|f\| = \|u\|$.

Proof. If f is complex linear and $u = \mathrm{Re} f$, then u is clearly real linear and $\mathrm{Im}[f(x)] = -\mathrm{Re}[if(x)] = -u(ix)$, so $f(x) = u(x) - iu(ix)$.

On the other hand, if u is real linear and $f(x) = u(x) - iu(ix)$, then f is clearly linear over $\mathbb{R}$, and $f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = if(x)$, so f is also linear over $\mathbb{C}$.

Finally, if X is normed, then $\|u\| \leq \|f\|$, since $|u(x)| = |\mathrm{Re} f(x)| \leq |f(x)|$. Also if $f(x) \neq 0$, let $\alpha = \overline{\mathrm{sgn} f(x)}$. Then $|f(x)| = \alpha f(x) = f(\alpha x) = u(\alpha x)$ (since $f(\alpha x)$ is real), so $|f(x)| \leq \|u\| \|\alpha x\|$, whence $\|f\| \leq \|u\|$. $\square$

Hahn–Banach theorem for complex normed vector spaces

Theorem (Hahn–Banach theorem for complex normed vector spaces)

Let E be a complex vector space and let p be a seminorm on E. If V is a subspace of E and if $f : V \to \mathbb{C}$ is a linear functional such that

$$|f(x)| \leq p(x) \quad \text{for} \quad x \in V,$$

then there is an extension $\hat{f} : E \to \mathbb{C}$ of f that is linear and satisfies

$$|\hat{f}(x)| \leq p(x) \quad \text{for} \quad x \in E,$$

Proof. Let $u = \mathrm{Re} f$. By the Hahn–Banach theorem for real normed vector spaces there is a real linear extension $\hat{u}$ of u to X such that $\hat{u}(x) \leq p(x)$ for all $x \in X$. Let $\hat{f}(x) = \hat{u}(x) - i\hat{u}(ix)$ as in the previous proposition. Then $\hat{f}$ is a complex linear extension of f, and if $\alpha = \overline{\mathrm{sgn} \hat{f}(x)}$ we have $|\hat{f}(x)| = \alpha \hat{f}(x) = \hat{f}(\alpha x) = \hat{u}(\alpha x) \leq p(\alpha x) = p(x)$. $\square$

Theorem

Let $(X, \|\cdot\|)$ be a normed vector space.

(a) *If M is a subspace of X and $f \in M^*$, then there exists $\hat{f} \in X^*$ an extension of f such that $\hat{f} = f$ on M and $\|\hat{f}\|_{X^*} = \|f\|_{M^*}$.*

(b) *If M is a closed subspace of X and $x \in X \setminus M$, there exists $f \in X^*$ such that $f(x) \neq 0$ and $f(y) = 0$ for every $y \in M$. In fact, f can be taken to satisfy $\|f\|_{X^*} = 1$ and $f(x) = \delta$, where $\delta = \inf_{y \in M} \|x - y\|$.*

(c) *If $0 \neq x \in X$, then there exists $f \in X^*$ such that $\|f\|_{X^*} = 1$ and $f(x) = \|x\|$.*

(d) *The bounded linear functionals on X separate points.*

(e) *If $x \in X$, define $\hat{x} : X^* \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. Then the map $x \mapsto \hat{x}$ is a linear isometry from X into X^{**} (the dual of X^*).*

Proof

Proof of (a). Define $p(x) = \|f\|_{M^*}\|x\|$ for $x \in X$. Then $|f| \le p$ on M. (Note that f is defined only on M.) By the Hahn–Banach extension theorem, there is $\hat{f} \in X^*$ such that $|\hat{f}| \le p$ on X and $\hat{f} = f$ on M. It follows directly that $\|\hat{f}\|_{X^*} = \|f\|_{M^*}$. Indeed, we have $\|\hat{f}\|_{X^*} \le \|f\|_{M^*}$, and also $\|f\|_{M^*} \le \|\hat{f}\|_{X^*}$, since $\hat{f}$ is an extension of f. $\square$

Proof of (b). Define f on $M + \mathbb{C}x$ by setting $f(y + \lambda x) = \lambda\delta$ for $y \in M$ and $\lambda \in \mathbb{C}$. Then $f(x) = \delta$, and $f = 0$ on M, and for $\lambda \ne 0$, we have $|f(y + \lambda x)| = |\lambda|\delta \le |\lambda|\|\lambda^{-1}y + x\| = \|y + \lambda x\|$. Thus, the Hahn–Banach theorem can be applied with $p(x) = \|x\|$ and M replaced by $M + \mathbb{C}x$. $\square$

Proof of (c). It is the special case of (b) with $M = \{0\}$. $\square$

Proof of (d). If $x \ne y$, then $f(x) \ne f(y)$, since by (c) there exists $f \in X^*$ with $f(x - y) = \|x - y\| \ne 0$. $\square$

Proof of (e). Obviously $\hat{x}$ is a linear functional on X^* and the map $x \mapsto \hat{x}$ is linear. Moreover, $|\hat{x}(f)| = |f(x)| \le \|f\|_{X^*}\|x\|$, so $\|\hat{x}\|_{X^*} \le \|x\|$. On the other hand, by (c) there exists $f \in X^*$ such that $f(x) = \|x\|$ thus $\|x\| = |f(x)| \le \|\hat{x}\|_{X^*}$. Hence $\|\hat{x}\|_{X^*} = \|x\|$ as desired. $\square$

Reflexive spaces

- With notation as in (e) above, let $\hat{X} = \{\hat{x} : x \in X\}$. Since X^{**} is always complete, the closure $\overline{\hat{X}}$ is a Banach space and the map $x \mapsto \hat{x}$ embeds X into $\overline{\hat{X}}$ as a dense subspace. $\overline{\hat{X}}$ is called the **completion** of X. In particular, if X is itself a Banach space, then $\overline{\hat{X}} = X$.
- A Banach space X is **reflexive** if $X = X^{**}$. For finite dimensional X we always have $X = X^{**}$ since these spaces have the same dimension. For infinite dimensional Banach spaces we may have $X \neq X^{**}$.
- Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If $1 < p < \infty$, then we show that $(L^p(X))^* = L^q(X)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In particular, $L^p(X)$ is reflexive for all $1 < p < \infty$. We also show that $(L^1(X))^* = L^\infty(X)$, but $(L^\infty(X))^* \neq L^1(X)$, when X is infinite.
- Let K be a compact Hausdorff topological space and $C(K)$ be the space of all continuous functions on K. It follows from Riesz theorem that $(C(K))^* = \mathcal{M}(K)$, where $\mathcal{M}(K)$ is the space of all complex Radon measures on K. It will be proved in 502!