Lecture 15

Hilbert spaces

Graduate Real Analysis

Fall Semester

Inner product

Definition (Inner product)

Let H be a complex vector space. The **inner product** (or **scalar product**) on H is a map

$$H \times H \ni (x, y) \mapsto \langle x, y \rangle \in \mathbb{C}$$

such that

1. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$ for all $x, y, x \in H$ and $a, b \in \mathbb{C}$,
2. $\overline{\langle x, y \rangle} = \langle y, x \rangle$ for all $x, y \in H$,
3. $\langle x, x \rangle \in (0, \infty)$ for all $x \in H \setminus \{0\}$.

Remark

Observe that (i) and (ii) imply

$$\langle x, ay + bz \rangle = \overline{a}\langle x, y \rangle + \overline{b}\langle x, z \rangle \quad \text{for all} \quad x, y, z \in H, \quad a, b \in \mathbb{C}.$$

Pre-Hilbert space

Real inner products

One can also define inner product on real vector space H: $\langle x, y \rangle$ is then real, one assumes that $a, b \in \mathbb{R}$ in (i), and (ii) becomes $\langle x, y \rangle = \langle y, x \rangle$.

Definition (Orthogonal elements)

We say that a vector $x \in H$ is **orthogonal** to $y \in H$ if

$$\langle x, y \rangle = 0,$$

and we write $x \perp y$.

Pre-Hilbert spaces

A complex vector space equipped with an inner product is called a **pre-Hilbert space.** If H is a pre-Hilbert space, for $x \in H$ we define

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

A useful fact

Proposition (A)

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. For $x, y \in H$ we have

$$x \perp y \quad \Longleftrightarrow \quad \|y\| \leq \|\lambda x + y\| \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

Proof ($\Longrightarrow$). If $x \perp y$ then $\alpha = \langle x, y\rangle = 0$, and

$$\begin{aligned} \|\lambda x + y\|^2 &= \langle \lambda x, \lambda x\rangle + \langle \lambda x, y\rangle + \langle y, \lambda x\rangle + \langle y, y\rangle \\ &= |\lambda|^2\|x\|^2 + 2\mathrm{Re}(\alpha\lambda) + \|y\|^2. \end{aligned}$$

Then we see that $\|\lambda x + y\|^2 = |\lambda|^2\|x\|^2 + \|y\|^2 \geq \|y\|^2$ as desired. □

Proof ($\Longleftarrow$). Let $\alpha = \langle x, y\rangle$ as before. If $x = 0$ then the claim is obvious. If $x \neq 0$ take $\lambda = -\frac{\overline{\alpha}}{\|x\|^2}$, and by the previous calculation, we have

$$\|y\|^2 \leq \|\lambda x + y\|^2 = |\lambda|^2\|x\|^2 + 2\mathrm{Re}(\alpha\lambda) + \|y\|^2 = \|y\|^2 - \frac{|\alpha|^2}{\|x\|^2},$$

which implies that $\alpha = \langle x, y\rangle = 0$. □

Cauchy–Schwarz inequality

Definition (The Cauchy–Schwarz inequality)

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. For all $x, y \in H$ we have

$$|\langle x, y\rangle| \leq \|x\|\|y\|$$

with equality iff x, y are linearly dependent.

Proof. If $\langle x, y\rangle = 0$ the result is obvious. If $\langle x, y\rangle \neq 0$, let $\alpha = \langle x, y\rangle$, and using calculations from the previous proposition with $\lambda = -\frac{\overline{\alpha}}{\|x\|^2}$ we obtain

$$\begin{aligned} 0 \leq \|\lambda x + y\|^2 &= |\lambda|^2\|x\|^2 + 2\mathrm{Re}(\alpha\lambda) + \|y\|^2 \\ &= \|y\|^2 - \frac{|\alpha|^2}{\|x\|^2}. \end{aligned}$$

Then

$$|\langle x, y\rangle| \leq \|x\|\|y\|. \qquad \square$$

Continuity of scalar products

Proposition (B)

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. If $x_n \xrightarrow[n\to\infty]{} x$, $y_n \xrightarrow[n\to\infty]{} y$ in H, then

$$\langle x_n, y_n\rangle \xrightarrow[n\to\infty]{} \langle x, y\rangle.$$

Proof. By the Cauchy–Schwarz inequality

$$\begin{aligned} |\langle x, y\rangle - \langle x_n, y_n\rangle| &\leq |\langle x_n - x, y_n\rangle| + |\langle x, y_n - y\rangle| \\ &\leq \|x - x_n\|\|y_n\| + \|x\|\|y - y_n\| \xrightarrow[n\to\infty]{} 0. \end{aligned}$$

□

Pre-Hilbert spaces and Hilbert spaces

Proposition

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. The function $H \ni x \mapsto \|x\| \in [0, \infty)$ is a **norm** *on H.*

Proof. $\|x\| = 0$ iff $x = 0$ and $\|\lambda x\| = |\lambda|\|x\|$. By the Cauchy–Schwarz inequality we obtain Minkowski's inequality inequality

$$\begin{aligned}
\|x + y\|^2 &= \langle x + y, x + y\rangle \\
&= \|x\|^2 + 2\mathrm{Re}(\langle x, y\rangle) + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.
\end{aligned}$$

□

Definition (Hilbert spaces)

A pre-Hilbert space $(H, \|\cdot\|)$ that is complete with respect to the norm is called a **Hilbert space**. Here $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in H$.

Hilbert spaces

Examples

1. $H = \mathbb{C}^d$ is a finite dimensional Hilbert space, with the scalar product

$$\langle x, y \rangle = \sum_{j=1}^{d} x_j \overline{y_j}.$$

2. Let A be a countable set. Then $H = \ell^2(A)$ is a Hilbert space, with the scalar product

$$\langle x, y \rangle = \sum_{j \in A} x_j \overline{y_j}.$$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $H = L^2(X)$, is a Hilbert space, with the scalar product

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} d\mu(x).$$

The Parallelogram Law

Proposition (The Parallelogram Law)

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. For all $x, y \in H$ we have

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof. Observe that

$$\|x \pm y\|^2 = \|x\|^2 \pm 2\mathrm{Re}(\langle x, y\rangle) + \|y\|^2.$$

Add these two formulas and we are done. □

Spaces with Parallelogram Law are Hilbert spaces

Theorem

If H is a normed vector space which satisfies the Parallelogram Law then there exists a unique inner product that gives rise to the norm.

Proof. If H is a real vector space the inner product is defined by

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right).$$

If H is a complex normed space the inner product is defined by

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right).$$

The Pythagorean Theorem

Definition

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. If $E \subseteq H$ we define

$$E^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in E\}.$$

It is easy to see by Proposition (B) that $E^\perp$ is a closed subspace of H.

Theorem (The Pythagorean Theorem)

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. If $x_1, \ldots, x_n \in H$ are $x_j \perp x_k$ for all $j \neq k$, then

$$\left\| \sum_{j=1}^n x_j \right\|^2 = \sum_{j=1}^n \|x_j\|^2.$$

Proof

Proof. We have

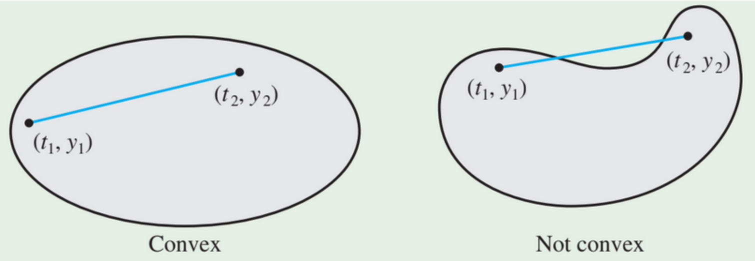
$$
\begin{aligned}
\left\| \sum_{j=1}^{n} x_j \right\|^2 &= \left\langle \sum_{j=1}^{n} x_j, \sum_{j=1}^{n} x_j \right\rangle \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \langle x_j, x_k \rangle \\
&= \sum_{j=1}^{n} \langle x_j, x_j \rangle + \sum_{j \neq k} \underbrace{\langle x_k, x_j \rangle}_{=0} \\
&= \sum_{j=1}^{n} \|x_j\|^2. \qquad \square
\end{aligned}
$$

Convex sets

Definition (Convex sets)

We say that $E \subseteq H$ is **convex** if for every $x, y \in E$ and $\lambda \in [0,1]$ we have

$$\lambda x + (1-\lambda)y \in E.$$



Best approximation result

Theorem (Best approximation theorem (BAT))

Let $(H, \|\cdot\|)$ be a Hilbert space. Every non-empty closed convex set $E \subseteq H$ contains a unique x of minimal norm.

Proof. Set $d = \inf\{\|x\| : x \in E\}$, and take a sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ such that $\|x_n\| \xrightarrow[n\to\infty]{} d$. Then by convexity

$$\frac{1}{2}(x_n + x_m) \in E,$$

which implies that $\|x_n + x_m\|^2 \geq 4d^2$ and by the parallelogram law we have

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2 \leq 2\|x_n\|^2 + 2\|x_m\|^2 - 4d^2 \xrightarrow[m,n\to\infty]{} 0.$$

So $(x_n)_{n\in\mathbb{N}}$ is a **Cauchy sequence** in H and E is closed, so it converges to some $x \in E$ with $\|x\| = d$. Moreover, $x \in E$ is unique. Show this! $\square$

Closed subspaces in Hilbert spaces are complemented

Theorem

Let $(H, \|\cdot\|)$ be a Hilbert space. If M is a closed subspace of H, then

$$H = M \oplus M^\perp \quad \text{and} \quad M \cap M^\perp = \{0\}.$$

$M^\perp$ is also closed and it is called the **orthogonal complement** *of M.*

Proof. If $x \in M \cap M^\perp$, then $\langle x, x\rangle = 0$, and $x = 0$, hence $M \cap M^\perp = \{0\}$. If $x \in H$ we apply **(BAT)** to the set $x - M$, which is convex and closed and find $P_M x \in M$ such that $\|x - P_M x\|$ is minimal. Set $z = x - P_M x$, then

$$\|z\| = \|x - P_M x\| \leq \|x - (P_M x + \lambda y)\| = \|z + \lambda y\|$$

for all $\lambda \in \mathbb{C}$ and $y \in M$, thus $z \perp y$ for any $y \in M$ (by Proposition (A)) hence $z \in M^\perp$. Since $x = P_M x + z$ we have $H \subseteq M \oplus M^\perp$, which gives $H = M \oplus M^\perp$. □

Corollary

Corollary

Let $(H, \|\cdot\|)$ be a Hilbert space. If M is a closed subspace of H, then

$$(M^{\perp})^{\perp} = M.$$

Proof. Take $x \in M$, then $\langle x, y\rangle = 0$ for all $y \in M^{\perp}$. But

$$(M^{\perp})^{\perp} = \{x \in H : \langle x, y\rangle = 0 \;\text{ for all }\; y \in M^{\perp}\},$$

thus $M \subseteq (M^{\perp})^{\perp}$. Moreover,

$$M \oplus M^{\perp} = H = M^{\perp} \oplus (M^{\perp})^{\perp},$$

so by the uniqueness of these decompositions M cannot be a proper subspace of $(M^{\perp})^{\perp}$. $\square$

Projections

Definition (Projections)

Let $(H, \|\cdot\|)$ be a Hilbert space and $M \subseteq H$ be its closed subspace. For every vector $x \in H$ we define its projection $P_M x$ onto M as the unique element of M with the property that

$$x - P_M x \perp M \quad \text{and} \quad \|x - P_M x\| \quad \text{is minimal.}$$

Proposition (C)

Let $(H, \|\cdot\|)$ be a Hilbert space with a closed subspace $M \subseteq H$. If a map $P_M : H \to H$ is defined by $P_M(x) = P_M x$ for all $x \in H$, then P_M defines a linear projection with $\|P_M\| = 1$.

Proof. To prove linearity, let $x, y \in H$, $\alpha, \beta \in \mathbb{C}$ and $z \in M$. Then

$$\begin{aligned}
&\langle \alpha x + \beta y - (\alpha P_M x + \beta P_M y), z \rangle \\
&= \alpha \langle x - P_M x, z \rangle + \beta \langle y - P_M y, z \rangle = 0.
\end{aligned}$$

Proof

Thus $\alpha P_M x + \beta P_M y \in M$ and $\alpha x + \beta y - (\alpha P_M x + \beta P_M y) \perp M$. But $P_M(\alpha x + \beta y)$ is the unique element with this property, so

$$P_M(\alpha x + \beta y) = \alpha P_M x + \beta P_M y.$$

For the second part, let $x \in H$ and $P_M x \in M$, so $x - P_M x \perp P_M x$. Then by the Pythagorean theorem

$$\|x\|^2 = \|P_M x\|^2 + \|x - P_M x\|^2 \geq \|P_M x\|^2,$$

hence $\|P_M x\| \leq \|x\|$, and

$$\|P_M\| = \sup_{\|x\|=1} \|P_M x\| = \sup_{\|x\|\leq 1} \|P_M x\| \leq 1.$$

If $x \in M$, then $P_M x = x$, so $\|P_M\| = 1$. This completes the proof. $\square$

Riesz representation theorem

Theorem (Riesz representation theorem)

Let $(H, \|\cdot\|)$ be a Hilbert space. If $\Lambda \in H^*$ then there is $y \in H$ such that $\Lambda x = \langle x, y \rangle$ for all $x \in H$. Moreover, we have $\|\Lambda\|_{H^*} = \|y\|$.

Remark

- The significance of this theorem is the identification of H^* with H. If H is a real Hilbert space, then H^* and H are isometrically isomorphic.
- If H is complex, it remains an isometry, but the correspondence is given by what is sometimes called a **conjugate isomorphism**. Specifically, if $\varphi, \psi \in H^*$ correspond to $u, v \in H$ (respectively), then $\alpha\varphi + \beta\psi$ corresponds to $\alpha u + \beta v$. Sometimes this is called **conjugate linearity** or **anti-isomorphism**.
- Regardless of the underlying scalar field (real or complex), a consequence of the identification of H with H^* is that any Hilbert space is reflexive.

Proof

Proof of uniqueness. Suppose that there are two $y, y' \in H$ such that $\Lambda(x) = \langle x, y \rangle = \langle x, y' \rangle$, then

$$\langle x, y - y' \rangle = 0 \quad \text{for all} \quad x \in H.$$

Thus

$$\|y - y'\|^2 = 0, \quad \text{so} \quad y = y'.$$

Proof of the existence. We first show that $\|\Lambda\|_{H^*} = \|y\|$. If $y \in H$ and $\Lambda x = \langle x, y \rangle$ for all $x \in H$, then

$$\|y\|^2 = \langle y, y \rangle = \Lambda y \leq \|\Lambda\|_{H^*} \|y\|,$$

so $\|y\| \leq \|\Lambda\|_{H^*}$. By the Cauchy–Schwarz we always have $|\Lambda x| \leq \|x\| \|y\|$ hence $\|\Lambda\|_{H^*} \leq \|y\|$ and consequently $\|\Lambda\|_{H^*} = \|y\|$.

Finally, we find $y \in H$ such that $\Lambda x = \langle x, y \rangle$ for all $x \in H$. If $\Lambda = 0$ take $y = 0$. If $\Lambda \neq 0$ let

$$\ker \Lambda = \{h \in H : \Lambda h = 0\}.$$

Proof

Since $\ker\Lambda$ is closed, we may write

$$H = (\ker\Lambda) \oplus (\ker\Lambda)^\perp$$

and $\Lambda \neq 0$ so there is $z \neq 0$ and $z \in (\ker\Lambda)^\perp$. We have

$$(\Lambda x)z - (\Lambda z)x \in \ker\Lambda \quad \text{for any} \quad x \in H,$$

hence $(\Lambda x)z - (\Lambda z)x \perp z$, and

$$\Lambda x\langle z, z\rangle = \Lambda z\langle x, z\rangle \quad \text{for any} \quad x \in H.$$

Then we see

$$\Lambda x = \left\langle x, \frac{(\overline{\Lambda z})z}{\langle z, z\rangle} \right\rangle.$$

It suffices to take

$$y = \frac{(\overline{\Lambda z})z}{\|z\|^2}$$

and we are done. $\square$

Orthonormal set and the Gram–Schmidt process

Definition (Orthonormal set)

A subset $\{u_\alpha : \alpha \in A\}$ of H is called **orthonormal** iff $\|u_\alpha\| = 1$ for all $\alpha \in A$ and $u_\alpha \perp u_\beta$ whenever $\alpha \neq \beta$.

Remark

If $(x_n)_{n\in\mathbb{N}}$ is a linearly independent sequence in H there is a standard inductive procedure, called the **Gram–Schmidt process** for converging $(x_n)_{n\in\mathbb{N}}$ into an orthonormal sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\mathrm{span}\{x_1, \ldots, x_n\} = \mathrm{span}\{u_1, \ldots, u_n\} \quad \text{for each} \quad n \in \mathbb{N},$$

and

$$u_i \perp u_j \quad \text{for} \quad i \neq j \quad \text{and} \quad 1 \leq i, j \leq n.$$

Gram–Schmidt algorithm

Step 1. Take $u_1 = \frac{x_1}{\|x_1\|}$.

Step 2. Having defined $u_1, u_2, \ldots, u_{N-1}$ we set

$$v_N = x_N - \sum_{j=1}^{N-1} \langle x_N, u_j \rangle u_j.$$

Then $v_N \neq 0$ since $x_N \notin \mathrm{span}\{x_1, \ldots, x_{N-1}\} = \mathrm{span}\{u_1, \ldots, u_{N-1}\}$ and for every $1 \leq m \leq N-1$ we have

$$\begin{aligned} \langle v_N, u_m \rangle &= \langle x_N, u_m \rangle - \sum_{j=1}^{N-1} \langle x_N, u_j \rangle \langle u_j, u_m \rangle \\ &= \langle x_N, u_m \rangle - \langle x_N, u_m \rangle \langle u_m, u_m \rangle = 0 \end{aligned}$$

since $\langle u_m, u_m \rangle = 1$. We can thus take

$$u_N = \frac{v_N}{\|v_N\|}.$$

Bessel's inequality

Theorem (Bessel's inequality)

Let $(H, \|\cdot\|)$ be a pre-Hilbert space. If $\{u_\alpha : \alpha \in A\}$ is an orthonormal system in H, then for any $x \in H$ we have

$$\sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2.$$

In particular, the set

$$\{\alpha \in A : \langle x, u_\alpha \rangle \neq 0\}$$

is countable.

Convention

$$\sum_{x \in X} f(x) = \sup\Big\{ \sum_{x \in F} f(x) : F \subseteq X, \quad \#F < \infty \Big\}.$$

Proof

Proof. It suffices to show

$$\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \leq \|x\|^2$$

for any finite $F \subseteq X$. We have

$$\begin{aligned}
\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 &= \sum_{\alpha \in F} \langle x, u_\alpha \rangle \overline{\langle x, u_\alpha \rangle} \\
&= \sum_{\alpha \in F} \langle x, \langle x, u_\alpha \rangle u_\alpha \rangle \\
&= \left\langle x, \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \right\rangle \\
&\leq \|x\| \left\| \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \right\|,
\end{aligned}$$

where in the last line we have used the Cauchy–Schwarz inequality.

Proof

Moreover, since $u_\alpha \perp u_\beta$ we obtain

$$\Big\| \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \Big\| = \Big(\sum_{\alpha,\beta \in F} \langle x, u_\alpha \rangle \langle u_\alpha, u_\beta \rangle \overline{\langle x, u_\beta \rangle} \Big)^{1/2}$$

$$= \Big(\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \Big)^{1/2}.$$

Hence

$$\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \leq \|x\| \Big(\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \Big)^{1/2}$$

and we are done since

$$\Big(\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 \Big)^{1/2} \leq \|x\|. \qquad \square$$

Orthonormal bases in Hilbert spaces

Theorem

If $\{u_\alpha : \alpha \in A\}$ is an orthonormal set in a Hilbert space $(H, \|\cdot\|)$ the following are equivalent:

(a) **Completeness:** *If $\langle x, u_\alpha \rangle = 0$ for all $\alpha \in A$, then $x = 0$.*

(b) **Parseval's identity:**

$$\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 \quad \text{for all} \quad x \in H.$$

(c) *For each $x \in H$ we have*

$$x = \sum_{\alpha \in A} \langle x, u_\alpha \rangle u_\alpha,$$

where the sum has only countably non-zero terms and converges in the norm topology no matters how these terms are ordered.

Proof

Proof (a) $\Longrightarrow$ (c). If $x \in H$ let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be enumeration of the α's for which $\langle x, u_\alpha \rangle \neq 0$. By Bessel's inequality, we have

$$\sum_{j \in \mathbb{N}} |\langle x, u_{\alpha_j} \rangle|^2 \leq \|x\|^2,$$

thus the series converges. By the Pythagorean theorem

$$\Big\| \sum_{j=m}^{n} \langle x, u_{\alpha_j} \rangle u_{\alpha_j} \Big\|^2 = \sum_{j=m}^{n} |\langle x, u_{\alpha_j} \rangle|^2 \xrightarrow[m,n \to \infty]{} 0$$

so the sum $\sum_{j=1}^{n} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$ converges, since H is complete.

Now if

$$y = x - \sum_{j \in \mathbb{N}} \langle x, u_{\alpha_j} \rangle u_{\alpha_j}$$

clearly $\langle y, u_{\alpha_j} \rangle = 0$ for all $\alpha \in A$ so by (a) we have $y = 0$. $\square$

Proof

Proof (c) $\implies$ (b). Take $F \subseteq A$ countable such that

$$x = \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha,$$

then

$$\sum_{\alpha \in F} |\langle x, u_\alpha \rangle|^2 = \left\langle x, \sum_{\alpha \in F} \langle x, u_\alpha \rangle u_\alpha \right\rangle = \|x\|^2. \qquad \square$$

Proof (b) $\implies$ (a). If $\langle x, u_\alpha \rangle = 0$ for all $\alpha \in A$ then $\|x\| = 0$, since

$$\|x\|^2 = \sum_{\alpha \in A} |\langle x, u_\alpha \rangle|^2 = 0. \qquad \square$$

Definition (Orthonormal basis)

An orthonormal set having the properties (a)-(c) is the previous theorem is called an **orthonormal basis** for H.

Orthonormal bases always exist for Hilbert spaces

Example

Let $H = \ell^2(A)$. For each $\alpha \in A$ define

$$e_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{e_\alpha : \alpha \in A\}$ is an orthonormal basis for $\ell^2(A)$ and $\langle f, e_\alpha \rangle = f(\alpha)$ for any $f \in \ell^2(A)$.

Theorem

Every Hilbert space $(H, \|\cdot\|)$ has an orthonormal basis.

Proof. By Kuratowski–Zorn's lemma the collection of all orthonormal sets ordered by inclusion has a maximal element and maximality is equivalent to the completeness property from item (a) of the previous teorem. $\square$

Orthogonal projections

Proposition (Orthogonal projections)

Let M be a closed subset of a Hilbert space $(H, \|\cdot\|)$. Suppose that a set $\{u_1, \ldots, u_N\}$ is an orthonormal basis for M, where $N \in \mathbb{N} \cup \{\infty\}$. For any $x \in H$ its orthogonal projection can be written by

$$P_M x = \sum_{n=1}^{N} \langle x, u_n \rangle u_n.$$

Proof. $P_M x \in M$ and $x - P_M x \perp M$. Thus

$$P_M x = \sum_{n=1}^{N} \langle P_M x, u_n \rangle u_n = \sum_{n=1}^{N} \langle P_M x - x, u_n \rangle u_n + \sum_{n=1}^{N} \langle x, u_n \rangle u_n. \qquad \square$$

Separable Hilbert spaces

Theorem

A Hilbert space $(H, \|\cdot\|)$ is separable iff it has a countable orthogonal basis, in which case every orthonormal basis for H is countable.

Proof ($\Longrightarrow$). If $\{x_n : n \in \mathbb{N}\}$ is a dense set in H, then by discarding inductively any x_n that is in a linear span of $x_1, \ldots, x_{n-1}$ we obtain a linearly independent sequence $\{y_n : n \in \mathbb{N}\}$ whose linear span is dense in H. Application of the Gram–Schmidt process to $\{y_n : n \in \mathbb{N}\}$ yields an orthonormal sequence $\{u_n : n \in \mathbb{N}\}$ whose linear span is dense in H and which is therefore a basis.

Proof ($\Longleftarrow$). If $\{u_n : n \in \mathbb{N}\}$ is a countable orthonormal basis in H the finite linear combinations of the u_n's with coefficients in a countable dense subset of $\mathbb{C}$ form a countable dense set in H. If $\{v_\alpha : \alpha \in A\}$ is another orthonormal basis, for each $n \in \mathbb{N}$ the set $A_n = \{\alpha \in A : \langle u_n, v_\alpha \rangle \neq 0\}$ is countable. By completeness of $\{u_n : n \in \mathbb{N}\}$, the set $A = \bigcup_{n \in \mathbb{N}} A_n$ is also countable. $\square$

Fourier basis

The set $\{e^{2\pi inx} : n \in \mathbb{Z}\}$, which is called the **standard Fourier basis**, is an orthonormal basis in $L^2([0,1])$. We have

$$\langle e^{2\pi inx}, e^{2\pi imx} \rangle = \int_0^1 e^{2\pi i(n-m)x} dx = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{otherwise.} \end{cases}$$

By the Weierstrass theorem every continuous function on $[0,1]$ can be uniformly approximated by trigonometric polynomials. In other words, $\overline{\mathcal{P}}^{\|\cdot\|_\infty} = C([0,1])$, where

$$\mathcal{P} = \Big\{ P(x) = \sum_{n=-N}^{N} a_n e^{2\pi inx} : N \in \mathbb{N} \text{ and } a_{-N}, \dots, a_N \in \mathbb{C} \Big\}.$$

Thus $\overline{\mathcal{P}}^{\|\cdot\|_{L^2}} = L^2([0,1])$, which proves, by the previous theorem, that $\{e^{2\pi inx} : n \in \mathbb{Z}\}$ is an orthonormal basis in $L^2([0,1])$.

Applications

Consider $f(x) = x$ on $[0,1]$, then

$$\langle f, 1 \rangle = \int_0^1 x dx = \frac{1}{2},$$

and

$$\begin{aligned}
\langle f, e^{2\pi inx} \rangle &= \int_0^1 x e^{-2\pi inx} dx \\
&= \int_0^1 x \left(\frac{e^{-2\pi inx}}{-2\pi in} \right)' dx \\
&= -\frac{1}{2\pi in} + \frac{1}{2\pi in} \int_0^1 e^{-2\pi inx} dx \\
&= -\frac{1}{2\pi in} + \frac{1}{2\pi in} \cdot 0 = \frac{i}{2\pi n}.
\end{aligned}$$

Example - application

By Parseval's identity we have

$$\frac{1}{3} = \|f\|_{L^2}^2 = \sum_{n\in\mathbb{N}} |\langle f, e^{2\pi inx}\rangle|^2$$
$$= \frac{1}{4} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{4\pi^2 n^2}.$$

Hence

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$

and finally we conclude

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$