Lecture 16

Operators in Hilbert spaces
von Neumann ergodic theorem
van der Corput's trick in Hilbert spaces

Graduate Real Analysis

Fall Semester

Bounded operators in Hilbert spaces

Definition

Let $B(H)$ denote the Banach space of all bounded linear operators on a Hilbert space $H \neq \{0\}$, with the operator norm

$$\|T\| = \|T\|_{H \to H} = \sup\{\|Tx\| : x \in H \text{ and } \|x\| \leq 1\}.$$

Proposition (A)

If $T \in B(H)$ and if $\langle Tx, x\rangle = 0$ for every $x \in H$ then $T = 0$. In particular, if $S \in B(H)$ and $T \in B(H)$ and

$$\langle Sx, x\rangle = \langle Tx, x\rangle$$

for all $x \in H$, then $S = T$.

Remark

This proposition fails if the scalar field is $\mathbb{R}$.

Proof

Proof. Since $\langle T(x+y), x+y\rangle = 0$ we see that

$$\langle Tx, y\rangle + \langle Ty, x\rangle = 0 \quad \text{for all} \quad x, y \in H.$$

If y is replaced by iy, then

$$-i\langle Tx, y\rangle + i\langle Ty, x\rangle = 0 \quad \text{for all} \quad x, y \in H.$$

Multiplying by i we have

$$\langle Tx, y\rangle - \langle Ty, x\rangle = 0.$$

Thus

$$2\langle Tx, y\rangle = 0,$$

so taking $y = Tx$ we obtain $\|Tx\|^2 = 0$ hence $Tx = 0$. □

Antilinear forms

Theorem (B)

Let $(H, \|\cdot\|)$ be a Hilbert space. If $f : H \times H \to \mathbb{C}$ is a antilinear form, i.e.

1. (a) $f(\alpha x + \beta y, z) = \alpha f(x,z) + \beta f(y,z)$,
2. (b) $f(x, \alpha y + \beta z) = \overline{\alpha} f(x,y) + \overline{\beta} f(x,z)$,
3. (c) *f is bounded in the sense*

$$M = \sup\{|f(x,y)| : \|x\| = \|y\| = 1\} < \infty.$$

Then there is a unique $S \in B(H)$ such that

$$f(x,y) = \langle x, Sy \rangle \quad \textit{and} \quad \|S\| = M.$$

Proof

Proof. Since

$$|f(x,y)| \leq M\|x\|\|y\| \quad \text{for all} \quad x,y \in H,$$

the mapping

$$x \mapsto f(x,y) \quad \text{for each fixed} \quad y \in H,$$

is a bounded linear functional on H with norm at most $M\|y\|$.

From the Riesz representation theorem it follows that for each $y \in H$ we find a unique element $Sy \in H$ such that

$$f(x,y) = \langle x, Sy \rangle,$$

also

$$\|Sy\| \leq M\|y\| \quad \text{for all} \quad y \in H.$$

Proof

It is easy to see that $S : H \to H$ is linear

$$\langle x, S(\alpha y)\rangle = f(x, \alpha y) = \overline{\alpha} f(x, y) = \overline{\alpha}\langle x, Sy\rangle = \langle x, \alpha S(y)\rangle$$

so $S(\alpha y) = \alpha S(y)$ by Proposition (A), and

$$\begin{aligned}\langle x, S(y_1 + y_2)\rangle = f(x, y_1 + y_2) =& f(x, y_1) + f(x, y_2)\\ &= \langle x, Sy_1\rangle + \langle x, Sy_2\rangle = \langle x, Sy_1 + Sy_2\rangle\end{aligned}$$

so $S(y_1 + y_2) = Sy_1 + Sy_2$, again by Proposition (A). Thus $S \in B(H)$ and we have $\|S\| \leq M$. Moreover, we also have

$$|f(x, y)| = |\langle x, Sy\rangle| \leq \|x\|\|Sy\| \leq \|x\|\|S\|\|y\|,$$

which implies that $M \leq \|S\|$, so we are done. $\square$

Adjoint operators

Fact

Let $(H, \|\cdot\|)$ be a Hilbert space. If $T \in B(H)$ then $\langle Tx, y\rangle$ is linear in x and antilinear in y and bounded

$$\Lambda(x, y) = \langle Tx, y\rangle, \quad \text{for all} \quad \|\Lambda\| = \|T\|.$$

Proof. Indeed, $|\langle Tx, y\rangle| \leq \|T\|\|x\|\|y\|$ so $\|\Lambda\| \leq \|T\|$ and also

$$|\langle Tx, Tx\rangle| = \|Tx\|^2$$

hence $\sup_{\|x\|=1} \|Tx\| \leq \|\Lambda\|$, and $\|T\| \leq \|\Lambda\|$.

Definition

By the previous theorem there is a unique $T^* \in B(H)$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{and} \quad \|T\| = \|T^*\|.$$

The operator T^* is called the **adjoint operator** to T.

Adjoint operators

Proposition

Let $(H, \|\cdot\|)$ be a Hilbert space. If $T, S \in B(H)$, then

$$(T+S)^* = S^* + T^*,$$
$$(\alpha T)^* = \overline{\alpha} T^*,$$
$$(ST)^* = T^* S^*,$$
$$T^{**} = T.$$

Moreover, we have

$$\|T\|^2_{H\to H} = \|T^* T\|_{H\to H}.$$

Remark

- The last identity is very important in the so-called T^*T methods of bounding linear operators in Hilbert spaces.

Proof

Proof. We only prove the second part, as the first part easily follows from the definition of adjoint operators and Proposition (A).

- We have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*T\|_{H\to H}\|x\|^2,$$

so

$$\|T\|_{H\to H}^2 \leq \|T^*T\|_{H\to H}.$$

- On the other hand,

$$\|T^*T\|_{H\to H} \leq \|T^*\|_{H\to H}\|T\|_{H\to H} = \|T\|_{H\to H}^2,$$

so

$$\|T\|_{H\to H}^2 = \|T^*T\|_{H\to H}.$$

This completes the proof. □

Kernel and range of linear operators

If $T \in B(H)$ then the kernel and range of T is defined, respectively, by

$$N(T) = \ker(T) = \{x \in H : Tx = 0\},$$
$$R(T) = \mathrm{range}(T) = \{y \in H : y = Tx \text{ for some } x \in H\}.$$

Theorem (C)

Let $(H, \|\cdot\|)$ be a Hilbert space. If $T \in B(H)$ then the

$$N(T^*) = R(T)^\perp \quad \textit{and} \quad N(T) = R(T^*)^\perp.$$

Proof. Note that

$$T^*y = 0 \iff \langle x, T^*y\rangle = 0 \quad \text{for all} \quad x \in H \iff y \in R(T)^\perp$$

Thus $N(T^*) = R(T)^\perp$ and $N(T^{**}) = R(T^*)^\perp$, since $T^{**} = T$. $\square$

Special operators in Hilbert spaces

Definition (Unitary operators)

A linear map $U : H_1 \to H_2$ between two Hilbert spaces $(H_1, \langle \cdot, \cdot \rangle_{H_1})$ and $(H_2, \langle \cdot, \cdot \rangle_{H_2})$ is called **unitary** or **orthogonal** if it is invertible and if

$$\langle Ux, Uy \rangle_{H_2} = \langle x, y \rangle_{H_1} \quad \text{for all} \quad x, y \in H_1.$$

Thus, a unitary operator is one-to-one and onto, and preserves the inner product.

Definition

Let $(H, \|\cdot\|)$ be a Hilbert space. An operator $T \in B(H)$ is said to be:

- **normal** if $T^*T = TT^*$,
- **self-adjoint** if $T = T^*$,
- a **projection** if $P^2 = P$.

Unitary operators on Hilbert spaces

Theorem (D)

Let $(H, \|\cdot\|)$ be a Hilbert space. If $U \in B(H)$ the following are equivalent:

- (a) *U is unitary,*
- (b) *$U^*U = UU^* = I$, where $I : H \to H$ is the identity operator on H,*
- (c) *$R(U) = H$ and $\|Ux\| = \|x\|$ for all $x \in H$.*

Proof (c) $\implies$ (b). If (c) holds, then

$$\langle U^*Ux, x\rangle = \langle Ux, Ux\rangle = \|Ux\|^2 = \|x\|^2 = \langle x, x\rangle, \quad \text{for all} \quad x \in H.$$

Hence $U^*U = I$. But (c) also implies that U is a linear isometry of H onto H so that U is invertible in $B(H)$ and $U^{-1} = U^*$, since $U^*U = I$.

Proof (b) $\implies$ (a). If $U^*U = UU^* = I$, then U is invertible and we have

$$\langle Ux, Uy\rangle = \langle x, U^*Uy\rangle = \langle x, y\rangle \quad \text{for all} \quad x, y \in H.$$

Proof (a) $\implies$ (c). This implication is obvious. $\square$

Universal Hilbert space

Theorem (E)

Let $(u_\alpha)_{\alpha\in A}$ be an orthonormal basis for $(H, \|\cdot\|)$, then the correspondence

$$x \mapsto \hat{x} \qquad \text{defined by} \qquad \hat{x}(\alpha) = \langle x, u_\alpha\rangle_H$$

is a unitary map from H to $\ell^2(A)$. Therefore, by this identification the $\ell^2(A)$ spaces are (sometimes) called universal Hilbert spaces.

Proof. We define $U : H \to \ell^2(H)$ by

$$U(x) = \hat{x} \quad \text{for} \quad x \in H.$$

This map is clearly linear and it is an isometry from H to $\ell^2(A)$, since by the Parseval identity we have

$$\|Ux\|^2_{\ell^2(A)} = \sum_{\alpha\in A} |\hat{x}(\alpha)|^2 = \|x\|^2_H.$$

Thus U is an isometry, and $\langle Ux, Uy\rangle_{\ell^2(A)} = \sum_{\alpha\in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)} = \langle x, y\rangle_H$.

Proof

We now prove that $R(U) = \ell^2(A)$. Indeed, if $f \in \ell^2(A)$, then

$$\sum_{\alpha\in A} |f(\alpha)|^2 < \infty.$$

The partial sums of the series

$$\sum_{\alpha\in A} f(\alpha)u_\alpha$$

are Cauchy. Hence

$$x_f = \sum_{\alpha\in A} f(\alpha)u_\alpha \in H,$$

and

$$U(x_f)(\alpha) = \hat{x}_f(\alpha) = \langle x_f, u_\alpha\rangle_H = f(\alpha) \quad \text{for all} \quad \alpha \in A.$$

So U is the isometry and onto, hence invertible and unitary as claimed. $\square$

Orthogonal projections

Theorem

Let $(H, \|\cdot\|)$ be a Hilbert space and let $P \in B(H)$ be a projection $P^2 = P$. Each of the following four properties of P implies the other three:

a. *P is self-adjoint, $P = P^*$,*
b. *P is normal,*
c. *$R(P) = N(P)^{\perp}$,*
d. *$\langle Px, x\rangle = \|Px\|^2$ for all $x \in H$.*

Property (c) is usually used to say that P is an orthogonal projection.

Proof

Proof $(a) \implies (b)$**.** Since $P = P^*$, then obviously $PP^* = P^*P$. $\square$

Proof $(b) \implies (c)$**.** If P is normal then

$$\|Px\|^2 = \langle Px, Px\rangle = \langle P^*Px, x\rangle = \langle PP^*x, x\rangle = \|P^*x\|^2.$$

so $N(P) = N(P^*)$ hence $N(P) = N(P^*) = R(P)^\perp$. Since P is a projection $R(P) = N(I - P)$ so that $R(P)$ is closed so

$$N(P)^\perp = R(P)^{\perp\perp} = R(P). \qquad \square$$

Proof $(c) \implies (d)$**.** For every $x \in H$ has the form $x = y + z$ with $y \perp z$, and $P(y) = 0$ and $P(z) = z$ and $\langle Px, x\rangle = \langle z, z\rangle$ so

$$\langle Px, x\rangle = \langle z, z\rangle = \|Pz\|^2 = \|Px\|^2. \qquad \square$$

Proof $(d) \implies (a)$**.** Note that

$$\|Px\|^2 = \langle Px, x\rangle = \langle x, P^*x\rangle = \langle P^*x, x\rangle.$$

Since $\|Px\|$ is real thus $P = P^*$. $\square$

von Neumann's ergodic theorem

Theorem (Mean Ergodic Theorem of von Neumann)

Let $(H, \|\cdot\|)$ be a Hilbert space. Let $U : H \to H$ be a unitary operator, then for any $f \in H$, we have

$$A_N f = \frac{1}{N} \sum_{n=0}^{N-1} U^n f \xrightarrow[N\to\infty]{} P_I f,$$

where P_I is the projection onto $I = \{x \in H : Ux = x\}$.

Proof. We begin with establishing Riesz decomposition

$$H = I \oplus \overline{R(U-I)}^H.$$

We claim that $I = R(U-I)^\perp$. Since U is unitary, thus normal, by Theorem (C), observe that

$$I = N(U-I) = N((U-I)^*) = R(U-I)^\perp.$$

Proof

Using Riesz's decomposition, any $f \in H$ can be written as $f = P_I h + h$, where $h \in \overline{R(U-I)}^H$. Note that $A_N P_I f = P_I f$. We claim that

$$A_N h \xrightarrow[N\to\infty]{} 0.$$

For $h \in R(U-I)$ so that $h = Ug - g$, by telescoping, we have

$$\|A_N h\| = \left\| \frac{1}{N} \sum_{n=0}^{N-1} U^n(Ug-g) \right\| = \frac{1}{N} \|U^N g - g\| \xrightarrow[N\to\infty]{} 0.$$

Let $(g_n)_{n\in\mathbb{N}} \subseteq H$ be such that $h_m \xrightarrow[m\to\infty]{} h$, where $h_m = Ug_m - g_m$. Thus

$$\|A_N h\| \leq \|A_N(h-h_m)\| + \|A_N h_m\| \leq \|h - h_m\| + \|A_N h_m\|.$$

Fix $\varepsilon > 0$ then there is $N_\varepsilon \in \mathbb{N}$ such that $\|h - h_m\| < \frac{\varepsilon}{2}$ and $\|A_N h_m\| < \frac{\varepsilon}{2}$ whenever $m, N \geq N_\varepsilon$, then $\|A_N h\| < \varepsilon$ for all $N \geq N_\varepsilon$ as desired. $\square$

van der Corput's inequality in Hilbert spaces

Theorem (van der Corput's inequality in Hilbert spaces)

Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in a Hilbert space $(H, \|\cdot\|)$. Define a sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers by

$$s_m = \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} \langle u_{n+m}, u_n \rangle \right|.$$

If

$$\lim_{H\to\infty} \frac{1}{H} \sum_{h=0}^{H-1} s_h = 0,$$

then

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} u_n \right\| = 0.$$

Proof

Proof. Assume that $M = \sup_{n\in\mathbb{N}} \|u_n\| < \infty$. Let $0 < H \leq N$, then

$$\frac{1}{N}\sum_{n=1}^{N} u_n = \frac{1}{N}\frac{1}{H}\sum_{h=1}^{H}\sum_{n\in\mathbb{Z}} u_{n+h}\mathbb{1}_{[1,N]}(n+h) =$$

$$\frac{1}{H}\sum_{h=1}^{H}\left[\frac{1}{N}\sum_{n=1}^{h} u_{n+h} - \frac{1}{N}\sum_{n=N+1}^{N+h} u_{n+h} + \frac{1}{N}\sum_{n\in\mathbb{Z}} u_{n+h}\mathbb{1}_{[h+1,N+h]}(n+h)\right].$$

By the triangle inequality

$$\left\|\frac{1}{N}\sum_{n=1}^{N} u_n\right\| \leq \frac{1}{NH}\left\|\sum_{h=1}^{H}\sum_{n\in[N]} u_{n+h}\right\| + \frac{2MH}{N},$$

since

$$\left\|\frac{1}{H}\frac{1}{N}\sum_{h=1}^{H}\sum_{n=1}^{h} u_{n+h}\right\| + \left\|\frac{1}{H}\frac{1}{N}\sum_{h=1}^{H}\sum_{n=N+1}^{N+h} u_{n+h}\right\| \leq \frac{2MH}{N}.$$

Proof

Let $[N] = [1, N]$ for $N \in \mathbb{N}$. By the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\left\| \frac{1}{N} \sum_{n \in [N]} \frac{1}{H} \sum_{h \in [H]} u_{n+h} \right\|^2 &\le \frac{1}{N} \sum_{n \in [N]} \left\| \frac{1}{H} \sum_{h \in [H]} u_{n+h} \right\|^2 \\
&= \left| \frac{1}{N} \sum_{n \in [N]} \frac{1}{H^2} \sum_{h_1, h_2 \in [H]} \langle u_{n+h_1}, u_{n+h_2} \rangle \right| \\
&\le \frac{2}{H^2} \sum_{\substack{h_1, h_2 \in [H] \\ h_2 \ge h_1}} \left| \frac{1}{N} \sum_{n \in [N] + h_1} \langle u_n, u_{n+h_2-h_1} \rangle \right| \\
&\le \frac{4M^2 H}{N} + \frac{2}{H^2} \sum_{\substack{h_1, h_2 \in [H] \\ h_2 \ge h_1}} \left| \frac{1}{N} \sum_{n \in [N]} \langle u_n, u_{n+h_2-h_1} \rangle \right| \\
&\le \frac{4M^2 H}{N} + \frac{2}{H^2} \sum_{h_1 \in [H]} \sum_{h \ge 0} \mathbb{1}_{[H]}(h + h_1) \left| \frac{1}{N} \sum_{n \in [N]} \langle u_n, u_{n+h} \rangle \right|.
\end{aligned}$$

Proof

Thus we proved that

$$\left\| \frac{1}{N} \sum_{n \in [N]} u_n \right\| \leq \left(\frac{2}{H} \sum_{h=0}^{H-1} \left| \frac{1}{N} \sum_{n \in [N]} \langle u_n, u_{n+h} \rangle \right| \right)^{1/2} + \frac{4MH^{1/2}}{N^{1/2}}.$$

Now, given $\varepsilon > 0$, we find $H_\varepsilon \in \mathbb{N}$ so that, for $H \geq H_\varepsilon$, we have

$$\frac{1}{H} \sum_{h=0}^{H-1} s_h < \varepsilon.$$

Then

$$\limsup_{N \to \infty} \left\| \frac{1}{N} \sum_{n \in [N]} u_n \right\| \leq \left(\frac{2}{H} \sum_{h=0}^{H-1} s_h \right)^{1/2} \leq (2\varepsilon)^{1/2}.$$

Letting $\varepsilon \to 0$ we conclude

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n \in [N]} u_n \right\| = 0. \qquad \square$$

van der Corput's trick

Remark

In fact, the proof gives a much stronger quantitative statement. Namely, for any $0 < H \leq N$, one has

$$\left\| \frac{1}{N} \sum_{n \in [N]} u_n \right\| \leq \left(\frac{2}{H} \sum_{h=0}^{H-1} \left| \frac{1}{N} \sum_{n \in [N]} \langle u_n, u_{n+h} \rangle \right| \right)^{1/2} + \frac{4H^{1/2}}{N^{1/2}} \sup_{n \in \mathbb{N}} \|u_n\|.$$