Lecture 17

Signed and complex measures and the Lebesgue–Radon–Nikodym theorem

Graduate Real Analysis

Fall Semester

Signed measures

Definition (Signed measures)

Let $(X, \mathcal{M})$ be a mesurable space. A **signed measure** on $(X, \mathcal{M})$ is a function $\nu : M \to [-\infty, \infty]$ such that

1. $\nu(\emptyset) = 0$,
2. ν assumes at most one of the values $\pm\infty$,
3. If $(E_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{M}$, then

$$\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \nu(E_n),$$

where the latter sum converges absolutely if $\nu\left(\bigcup_{n=1}^{\infty} E_n\right)$ is finite.

- Thus every measure is a signed measure; for emphasis we shall sometimes refer to measures as **positive measures.**

Examples of signed measures

Example 1

If μ_1, μ_2 are measures on $(X, \mathcal{M})$ and at least one of them is finite then $\nu = \mu_1 - \mu_2$ is a signed measure.

Example 2

If μ is a measure on $(X, \mathcal{M})$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of the integrals

$$\int_X f^+ d\mu \quad \text{and} \quad \int_X f^- d\mu$$

is finite (in this case we call f **an extended μ-integrable** function), then the function defined by

$$\nu(E) = \int_E f d\mu$$

is a signed measure.

Proposition

Remark

As we shall see later every signed measure can be represented in either of these two forms from the previous two examples.

Proposition

Let ν be a signed measure on $(X, \mathcal{M})$. If $(E_j)_{j\in\mathbb{N}}$ is an increasing sequence in $\mathcal{M}$, then

$$\nu\Big(\bigcup_{j=1}^{\infty} E_j\Big) = \lim_{j\to\infty} \nu(E_j).$$

If $(E_j)_{j\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then

$$\nu\Big(\bigcap_{j=1}^{\infty} E_j\Big) = \lim_{j\to\infty} \nu(E_j).$$

Proof. The same as for positive measures. □

Positive, negative, and null set for signed measures

Definition (Positive, negative and null sets)

Let ν be a signed measure on $(X, \mathcal{M})$.

- A set $E \in \mathcal{M}$ is called **positive** for ν if $\nu(F) \geq 0$ for all $F \in \mathcal{M}$ such that $F \subseteq E$.
- A set $E \in \mathcal{M}$ is called **negative** for ν if $\nu(F) \leq 0$ for all $F \in \mathcal{M}$ such that $F \subseteq E$.
- A set $E \in \mathcal{M}$ is called **null** for ν if $\nu(F) = 0$ for all $F \in \mathcal{M}$ and $F \subseteq E$.

Remark

If $\nu(E) = \int_E f d\mu$, then

- E is positive for ν if $f \geq 0$ μ-a.e. on E,
- E is negative for ν if $f \leq 0$ μ-a.e. on E,
- E is null for ν if $f = 0$ μ-a.e. on E.

Useful fact

Lemma

Any measurable subset of a positive set is positive and the union of any countable family of positive sets is positive.

Proof. The first part is clear from the definition of positivity. If $P_1, P_2, \ldots$ are positive sets, let

$$Q_n = P_n \setminus \bigcup_{j=1}^{n-1} P_j,$$

then $Q_n \subseteq P_n$ so Q_n is positive. Hence if

$$E \subseteq \bigcup_{j=1}^{\infty} P_j$$

then

$$\nu(E) = \sum_{n=1}^{\infty} \nu(Q_n \cap E) \geq 0. \qquad \square$$

Total variation

Problem

Given a signed measure ν on $(X, \mathcal{M})$ it is always possible to find a (positive) measure μ that dominates ν, in the sense that

$$\nu(E) \leq \mu(E) \quad \text{for all} \quad E \in \mathcal{M}$$

and in addition μ is the "smallest" one that has this property.

Definition (Total variation)

Let ν be a signed measure on $(X, \mathcal{M})$. For every $E \in \mathcal{M}$ we define a function $|\nu|$, called the **total variation** of ν, by setting

$$|\nu|(E) = \sup\left\{ \sum_{k=1}^{\infty} |\nu(E_k)| : (E_k)_{k\in\mathbb{N}} \subseteq \mathcal{M} \text{ is disjoint and } E = \bigcup_{k\in\mathbb{N}} E_k \right\}.$$

The supremum is taken over all measurable partitions $(E_k)_{k\in\mathbb{N}}$ of $E \in \mathcal{M}$.

Theorem

Theorem

The total variation $|\mu|$ of a signed measure μ on $(X, \mathcal{M})$ is itself a positive measure on $(X, \mathcal{M})$ and $\mu \leq |\mu|$.

Proof. Let $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ be a partition of $E \in \mathcal{M}$. Let $t_j < |\mu|(E_j)$. Then each E_j has a partition $(A_{ij})_{i\in\mathbb{N}}$ so that

$$\sum_{j\in\mathbb{N}} |\mu(A_{ij})| > t_i \quad \text{for all} \quad i \in \mathbb{N}.$$

Since $(A_{ij})_{i,j\in\mathbb{N}}$ is a partition of E it follows that

$$\sum_{i\in\mathbb{N}} t_i \leq \sum_{i,j\in\mathbb{N}} |\mu(A_{ij})| \leq |\mu|(E).$$

Taking the sup over all possible choices of $(t_i)_{i\in\mathbb{N}}$ on the left-hand-side of this inequality we obtain that $\sum_{i\in\mathbb{N}} |\mu|(E_i) \leq |\mu|(E)$.

Proof

To prove the opposite inequality let $(A_j)_{j\in\mathbb{N}}$ be any partition of $E \in \mathcal{M}$. Then for any fixed $j \in \mathbb{N}$, the sequence $(A_j \cap E_i)_{i\in\mathbb{N}}$ is a partition of A_j and for any $i \in \mathbb{N}$ the sequence $(A_j \cap E_i)_{j\in\mathbb{N}}$ is a partition of E_i. Hence,

$$\begin{aligned}\sum_{j\in\mathbb{N}} |\mu(A_j)| &= \sum_{j\in\mathbb{N}} \Big| \sum_{i\in\mathbb{N}} \mu(A_j \cap E_i) \Big| \le \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} |\mu(A_j \cap E_i)| \\ &= \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} |\mu(A_j \cap E_i)| \le \sum_{i\in\mathbb{N}} |\mu|(E_i),\end{aligned}$$

thus

$$|\mu|(E) \le \sum_{j\in\mathbb{N}} |\mu|(E_j),$$

since $(A_j)_{j\in\mathbb{N}}$ was an arbitrary partition of E. $\square$

Positive and negative variations

Definition (Positive and negative variations)

Given a signed measure ν on $(X, \mathcal{M})$ it is now possible to write ν as the difference of two (positive) measures. To see this, we define the **positive variation** and **negative variation** of ν by

$$\nu^+ = \frac{1}{2}(|\nu| + \nu) \quad \text{and} \quad \nu^- = \frac{1}{2}(|\nu| - \nu).$$

Remark

- We see that ν^+ and ν^- are positive measures, and

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad |\nu| = \nu^+ + \nu^-.$$

- If $\nu(E) = \infty$ for some $E \in \mathcal{M}$, then $|\nu|(E) = \infty$, and $\nu^-(E)$ is defined to be 0 in this case.
- One more piece of terminology: a signed measure ν is called **finite** (resp. σ-**finite**) if $|\nu|$ is finite (resp. σ-finite).

Mutually singular measures

Definition (Integration with respect to a signed measure)

Integration with respect to a signed measure ν on $(X, \mathcal{M})$ is defined by setting $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$ and $\int_X f d\nu = \int_X f d\nu^+ - \int_X f d\nu^-$.

Definition (Support of a measure)

Let μ be a positive or signed measure on $(X, \mathcal{M})$. We say that μ is **supported** on $A \in \mathcal{M}$ if $\mu(E) = \mu(E \cap A)$ for every $E \in \mathcal{M}$.

- Equivalently $\mu(E) = 0$ for every $E \in \mathcal{M}$ such that $E \cap A = \emptyset$.

Definition (Mutually singular measures)

We say that two signed measures μ and ν on $(X, \mathcal{M})$ are **mutually singular** or that ν **is singular with respect to** μ or vice versa if there exist $E, F \in \mathcal{M}$ such that $E \cap F = \emptyset$ and μ is supported on E and ν is supported on F. We write that $\mu \perp \nu$.

Absolutely continuous measures

Definition (Absolutely continuous measures)

Let ν be a signed measure and μ be a positive measure on $(X, \mathcal{M})$. We say that ν is **absolutely continuous** with respect to μ and write

$$\nu \ll \mu$$

if $\nu(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$.

Remark

One can easily verify that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Proof. Suppose that $\nu \ll \mu$ and take $E \in \mathcal{M}$ such that $\mu(E) = 0$. Take any disjoint partition $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ of E and note that $\mu(E_n) = 0$ for any $n \in \mathbb{N}$, thus $\sum_{n\in\mathbb{N}} |\nu(E_n)| = 0$, since $\nu \ll \mu$ and consequently $|\nu|(E) = 0$. If $|\nu| \ll \mu$, then $\nu \ll \mu$, since $\nu \leq |\nu|$. Finally, $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, since $|\nu| = \nu^+ + \nu^-$. $\square$

The ε-δ criterion for absolute continuity

Remark

Let ν be a signed measure and μ be a positive measure on $(X, \mathcal{M})$. If $\nu \perp \mu$ and $\nu \ll \mu$, then $\nu = 0$.

Proof. Let $E, F \in \mathcal{M}$ be such that $E \cap F = \emptyset$, and ν is supported on E and μ is supported on F. Then $\mu(E) = 0$, and $|\nu|(E) = 0$, since $\nu \ll \mu$ iff $|\nu| \ll \mu$. Hence $|\nu|(E) = 0$, and consequently $|\nu| = 0$ and $\nu = 0$. $\square$

Theorem (The ε-δ criterion for absolute continuity)

Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$, then $\nu \ll \mu$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) < \delta \quad \Longrightarrow \quad |\nu|(E) < \varepsilon.$$

Proof ($\Longleftarrow$). Since $\nu \ll \mu$ iff $|\nu| \ll \mu$ and $|\nu(E)| \leq |\nu|(E)$, so it suffices to assume that $\nu = |\nu|$ is positive. If $E \in \mathcal{M}$ and $\mu(E) = 0$, then the ε-δ condition implies that $\nu \ll \mu$. $\square$

Proof

Proof ($\Longrightarrow$). On the other hand if the ε-δ condition is not satisfied, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ we can find $E_n \in \mathcal{M}$ with

$$\mu(E_n) < 2^{-n} \quad \text{and} \quad \nu(E_n) \geq \varepsilon.$$

Let $F_k = \bigcup_{j=k}^{\infty} E_j$ and consider

$$F = \bigcap_{k=1}^{\infty} F_k.$$

Then $\mu(F) = 0$, since

$$\mu(F_k) < \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k}.$$

Since ν is finite and $\nu(F_k) \geq \varepsilon$ for all $k \in \mathbb{N}$, we have

$$\nu(F) = \lim_{k \to \infty} \nu(F_k) \geq \varepsilon > 0,$$

which is impossible, since we have assumed that $\nu \ll \mu$. $\square$

Corollary

Let $(X, \mathcal{M}, \mu)$ be a measure space. If μ is a measure and f is an extended μ-integrable function, the signed measure ν defined by

$$\nu(E) = \int_E f d\mu \quad \text{for} \quad E \in \mathcal{M}$$

is clearly absolutely continuous with respect to μ. It is also finite iff $f \in L^1(X, \mu)$. For any complex-valued $f \in L^1(X, \mu)$ the ε-δ criterion can be applied to $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ and we obtain the following corollary.

Corollary

Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f \in L^1(X, \mu)$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(E) < \delta \quad \Longrightarrow \quad \Big| \int_E f d\mu \Big| < \varepsilon.$$

We will write $d\nu = f d\mu$ to express the relationship $\nu(E) = \int_E f d\mu$.

Proposition

Proposition

Suppose that $(X, \mathcal{M}, \mu)$ is a finite measure space $\mu(X) < \infty$. Let $f \in L^1(X)$ and $S \subseteq \mathbb{C}$ be a closed set and

$$A_E f = \frac{1}{\mu(E)} \int_E f d\mu \in S$$

for every $E \in \mathcal{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

Proof. Note that $S^c = \bigcup_{n\in\mathbb{N}} B(\alpha_n, r_n)$, where $B(\alpha_n, r_n) \subseteq S^c$ is a closed disc centered at $\alpha_n \in \mathbb{C}$, with radius $r_n > 0$. Then it suffices to prove that $\mu(E_n) = 0$, with $E_n = f^{-1}[B(\alpha_n, r_n)]$ for $n \in \mathbb{N}$. If we had $\mu(E_n) > 0$ then

$$|A_{E_n}(f) - \alpha_n| = \frac{1}{\mu(E_n)} \left| \int_{E_n} (f - \alpha_n) d\mu \right| \leq \frac{1}{\mu(E_n)} \int_{E_n} |f - \alpha_n| d\mu \leq r_n$$

this is impossible since $A_E(f) \in S$. Hence $\mu(E) = 0$. □

The Lebesgue–Radon–Nikodym theorem

Theorem (The Lebesgue–Radon–Nikodym theorem)

Let ν be a σ-finite signed measure and μ be a σ-finite positive measure on $(X, \mathcal{M})$. Then the following are true.

(i) *(**Lebesgue decomposition**) There exist unique σ-finite signed measures ν_a and ν_s on $(X, \mathcal{M})$ such that*

$$\nu_a \ll \mu \quad \text{and} \quad \nu_s \perp \mu \quad \text{and} \quad \nu = \nu_a + \nu_s.$$

(ii) *(**Radon–Nikodym theorem**) Moreover, there is an extended μ-measurable function $f : X \to \mathbb{R}$ such that*

$$\nu_a(E) = \int_E f(x) d\mu(x) \quad \text{for} \quad E \in \mathcal{M}$$

and any two such functions are equal μ-a.e. on X.

Proof

Proof of the uniqueness. Suppose that we have another pair of σ-finite signed measures λ_a and λ_s on $(X, \mathcal{M})$ such that

$$\lambda_a \ll \mu \quad \text{and} \quad \lambda_s \perp \mu \quad \text{and} \quad \nu = \lambda_a + \lambda_s.$$

Then

$$\nu_a + \nu_s = \lambda_a + \lambda_s \quad \Longleftrightarrow \quad \lambda = \nu_a - \lambda_a = \lambda_s - \nu_s.$$

But $\lambda = \nu_a - \lambda_a \ll \mu$ and $\lambda = \lambda_s - \nu_s \perp \mu$, hence we must have $\lambda = 0$, which means that $\nu_a = \lambda_a$ and $\lambda_s = \nu_s$ and proves the uniqueness. $\square$

Proof of the existence. We shall present von Neumann's argument that exploits elegantly simple Hilbert space ideas.

- We start with the case when both ν and μ are positive and finite measures. Let $\lambda = \nu + \mu$ and consider

$$\Lambda(\varphi) = \int_X \varphi(x) d\nu(x) \quad \text{for} \quad \varphi \in L^2(X, \lambda).$$

Proof

- By the Cauchy–Schwarz inequality we see that the mapping Λ defines a bounded linear functional on $L^2(X, \lambda)$, namely

$$|\Lambda(\varphi)| \leq \int_X |\varphi(x)| d\nu(x) \leq \int_X |\varphi(x)| d\lambda(x) \leq \lambda(X)^{1/2} \|\varphi\|_{L^2(X,\lambda)}.$$

- By the Riesz representation theorem, since $L^2(X, \lambda)$ is a Hilbert space, we find $g \in L^2(X, \lambda)$ such that

$$\Lambda(\varphi) = \int_X \varphi(x) d\nu(x) = \int_X \varphi(x) g(x) d\lambda(x) \quad \text{for all} \quad \varphi \in L^2(X, \lambda).$$

- If $E \in \mathcal{M}$ and $\lambda(E) > 0$ and we take $\varphi = \mathbb{1}_E$ in the previous identity, and using $\nu \leq \lambda$ we obtain

$$0 \leq \frac{\Lambda(\mathbb{1}_E)}{\lambda(E)} = \frac{1}{\lambda(E)} \int_E g(x) d\lambda(x) \leq 1,$$

  which by the previous proposition gives $0 \leq g(x) \leq 1$ for λ-a.e $x \in X$.

Proof

- In fact, we can assume that $0 \leq g(x) \leq 1$ for every $x \in X$ without disturbing the identity

$$\int_X \varphi d\nu = \int_X \varphi g d\lambda \quad \text{for all} \quad \varphi \in L^2(X, \lambda),$$

which can be rewritten in the following form

$$\int_X \varphi(1-g) d\nu = \int_X \varphi g d\mu \quad \text{for all} \quad \varphi \in L^2(X, \lambda). \qquad (*)$$

- Consider now the two sets

$$A = \{x \in X : 0 \leq g(x) < 1\}, \quad \text{and} \quad B = \{x \in X : g(x) = 1\},$$

and define two measures ν_a and ν_s on $\mathcal{M}$ by

$$\nu_a(E) = \nu(A \cap E) \quad \text{and} \quad \nu_s(E) = \nu(B \cap E) \quad \text{for} \quad E \in \mathcal{M}.$$

Proof

- To see why $\nu_s \perp \mu$ we set $\varphi = \mathbb{1}_B$ in (*) and obtain

$$0 = \int_B (1-g)\,d\nu = \int_B g\,d\mu = \mu(B).$$

- Finally, we set $\varphi = \mathbb{1}_E(1+g+\ldots+g^n)$, in (*), then we obtain

$$\begin{aligned}\int_E (1-g^{n+1})\,d\nu &= \int_E (1+g+\ldots+g^n)(1-g)\,d\nu \\ &= \int_E g(1+g+\ldots+g^n)\,d\mu\end{aligned}$$

- Since $1-g^{n+1}(x) = 0$ if $x \in B$ and $1-g^{n+1}(x) \xrightarrow[n\to\infty]{} 1$ for $x \in A$, the **(DCT)** implies that

$$\lim_{n\to\infty} \int_E (1-g^{n+1})\,d\nu = \nu(A \cap E) = \nu_a(E).$$

Proof

- On the other hand, by the **(MCT)** we obtain

$$\nu_a(E) = \lim_{n\to\infty} \int_E g(1+g+\ldots+g^n)d\mu = \int_E f d\nu, \quad \text{where} \quad f = \frac{g}{1-g}.$$

- Observe also that $f \in L^1(X,\mu)$, since $\nu_a(X) \leq \nu(X) < \infty$.
- If μ and ν are σ-finite and positive we may clearly find a sequence $(E_j)_{j\in\mathbb{N}} \subseteq \mathcal{M}$ of disjoint sets such that $X = \bigcup_{j\in\mathbb{N}} E_j$ and

$$\mu(E_j) < \infty \quad \text{and} \quad \nu(E_j) < \infty \quad \text{for all} \quad j \in \mathbb{N}.$$

- For $E \in \mathcal{M}$ we may define positive and finite measures by

$$\mu_j(E) = \mu(E \cap E_j) \quad \text{and} \quad \nu_j(E) = \nu(E \cap E_j) \quad \text{for all} \quad j \in \mathbb{N}.$$

Proof

- Then for each $j \in \mathbb{N}$, by the previous part, we can write

$$\nu_j = \nu_{j,a} + \nu_{j,s}, \quad \text{where} \quad \nu_{j,a} \ll \mu_j, \quad \text{and} \quad \nu_{j,s} \perp \mu_j.$$

- Moreover, we have $\nu_{j,a} = f_j d\mu_j$ for some $f_j \in L^1(X, \mu_j)$.
- Then it suffices to set

$$f = \sum_{j \in \mathbb{N}} f_j, \quad \text{and} \quad \nu_a = \sum_{j \in \mathbb{N}} \nu_{j,a}, \quad \text{and} \quad \nu_s = \sum_{j \in \mathbb{N}} \nu_{j,s}.$$

- Finally, if ν is signed, then we apply the argument separately to the positive and negative variations of ν.

This completes the proof of the Lebesgue–Radon–Nikodym theorem. $\square$

Complex measures

Definition (Complex measures)

Let $(X, \mathcal{M})$ be a mesurable space. A **complex measure** on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to \mathbb{C}$ such that

(i) $\nu(\emptyset) = 0$,

(ii) If $(E_n)_{n\in\mathbb{N}}$ is a sequence of disjoint sets in $\mathcal{M}$, then

$$\nu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} \nu(E_n),$$

where the series converges absolutely.

Some remarks and properties

Remarks

- Infinite values are not allowed, so a positive measure is a complex measure only if it is finite. For example: if μ is positive measure and $f \in L^1(X, \mu)$ then $d\nu = fd\mu$ is a complex measure.
- If ν is a complex measure we shall write ν_r and ν_i for the real and imaginary parts of ν.
- Thus ν_r, ν_i are signed measures that do not assume values $\pm\infty$, hence they are finite and so the range of ν is a bounded subset of $\mathbb{C}$.
- We define $L^1(X, \nu) = L^1(X, \nu_r) \cap L^1(X, \nu_i)$ and for $f \in L^1(X, \nu)$ we set $\int_X fd\nu = \int_X fd\nu_r + i \int f_X d\nu_i$.
- If ν and μ are complex measures we say that $\nu \perp \mu$ if $\nu_a \perp \mu_b$ for $a, b \in \{r, i\}$.
- If λ is a positive measure we say $\nu \ll \lambda$ if $\nu_r \ll \lambda$ and $\nu_i \ll \lambda$.

Total variation for complex measures

Definition (Total variation for complex measures)

Let ν be a complex measure on $(X, \mathcal{M})$. For every $E \in \mathcal{M}$ we define a function $|\nu|$, called the **total variation** of ν, by setting

$$|\nu|(E) = \sup \left\{ \sum_{k=1}^{\infty} |\nu(E_k)| : (E_k)_{k\in\mathbb{N}} \subseteq \mathcal{M} \text{ is disjoint and } E = \bigcup_{k\in\mathbb{N}} E_k \right\}.$$

The supremum is taken over all measurable partitions $(E_k)_{k\in\mathbb{N}}$ of $E \in \mathcal{M}$.

Theorem (Total variation is a positive measure)

The total variation $|\nu|$ of a complex measure ν on $(X, \mathcal{M})$ is a positive measure on $(X, \mathcal{M})$.

Proof. The proof goes exactly the same way as for signed measures. □

Simple facts

Proposition

Let ν, ν_1, ν_2 be complex measures on $(X, \mathcal{M})$, and let μ be a positive measure on $(X, \mathcal{M})$.

1. *If ν is supported on A, so is $|\nu|$.*
2. *If $\nu_1 \perp \nu_2$, then $|\nu_1| \perp |\nu_2|$.*
3. *If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.*
4. *If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.*
5. *If $\nu \ll \mu$, then $|\nu| \ll \mu$.*
6. *If $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.*
7. *If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.*

Proof. It is an easy exercise. □

The Lebesgue–Radon–Nikodym theorem revised

Theorem (The Lebesgue–Radon–Nikodym theorem for complex measures)

Let ν be a complex measure and μ be a σ-finite positive measure on $(X, \mathcal{M})$. Then the following are true.

(i) *(**Lebesgue decomposition**) There exist unique complex measures ν_a and ν_s on $(X, \mathcal{M})$ such that*

$$\nu_a \ll \mu \quad \text{and} \quad \nu_s \perp \mu \quad \text{and} \quad \nu = \nu_a + \nu_s.$$

(ii) *(**Radon–Nikodym theorem**) Moreover, there is a unique $f \in L^1(X, \mu)$ such that*

$$\nu_a(E) = \int_E f(x) d\mu(x) \quad \text{for} \quad E \in \mathcal{M}.$$

Proof. It suffices to apply the Lebesgue–Radon–Nikodym theorem for signed measures to the real and imaginary parts of the measure ν. □